DIFFERENTIAL CALCULUS

FOR

BEGINNERS.

BY

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PREFACE.

THE present small volume is intended to form a sound introduction to a study of the Differential Calculus *suitable for the beginner*. It does not therefore aim at completeness, but rather at the omission of all portions which are usually considered best left for a later reading. At the same time it has been constructed to include those parts of the subject prescribed in Schedule I. of the Regulations for the Mathematical Tripos Examination for the reading of students for Mathematical Honours in the University of Cambridge.

Particular attention has been given to the examples which are freely interspersed throughout the text. For the most part they are of the simplest kind, requiring but little analytical skill. Yet it is hoped they will prove sufficient to give practice in the processes they are intended to illustrate.

It is assumed that in commencing to work at the Differential Calculus the student possesses a fair know-

ledge of Algebra as far as the Exponential and Logarithmic Theorems; of Trigonometry as far as Demoivre's Theorem, and of the rudiments of Cartesian Geometry as far as the equations of the several Conic Sections in their simplest forms.

Being to some extent an abbreviation of my larger Treatise my acknowledgments are due to the same authorities as there mentioned. My thanks are also due to several friends for useful suggestions with regard to the desirable scope of the book.

Any suggestions for its improvement or for its better adaptation to the requirements of junior students, or lists of errata, will be gratefully received.

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CONTENTS.

CHAPTER I.

Limiting Values. Elementary Undetermined Forms.

CHAPTER II.

Differentiation from the Definition.

CHAPTER III.

Fundamental Propositions.

CHAPTER IV.

Standard Forms.

CHAPTER V.

Successive Differentiation.

CHAPTER VI.

Expansions.

CHAPTER VII.

Infinitesimals.

CHAPTER VIII.

Tangents and Normals.

CHAPTER IX.

Asymptotes.

CHAPTER X.

Curvature.

CHAPTER XI.

Envelopes.

CHAPTER XII.

Associated Loci.

CHAPTER XIII.

Maxima and Minima.

CHAPTER XIV.

Undetermined Forms.

CHAPTER XV.

Limitations of Taylor's Theorem.

DIFFERENTIAL CALCULUS.

CHAPTER I.

LIMITING VALUES. ELEMENTARY UNDETERMINED FORMS.

1. **Object of the Differential Calculus.** When an increasing or decreasing quantity is made the subject of mathematical treatment, it often becomes necessary to estimate its rate of growth. It is our principal object to describe the method to be employed and to exhibit applications of the processes described.

2. **Explanation of Terms.** The frequently recurring terms "Constant," "Variable," "Function," will be understood from the following example:

Let the student imagine a triangle of which two sides x, y are unknown but of which the angle (A) included between those sides is known. The area (Δ) is expressed by

$$\Delta=\tfrac{1}{2}xy\sin A.$$

The quantity A is a "*constant*" for by hypothesis it retains the same value, though the sides x and y may change in length while the triangle is under observation. The quantities x, y and Δ are therefore called *variables*. Δ, whose value *depends upon* those of x and y, is called the *dependent variable*; x and y, whose values may be any whatever, and may either or both take up any values which may be assigned to them, are called *independent variables*.

The quantity Δ whose value thus depends upon those of x, y and A is said to be a *function* of x, y and A.

3. Definitions. We are thus led to the following definitions:

(*a*) *A* CONSTANT *is a quantity which, during any set of mathematical operations, retains the same value.*

(*b*) *A* VARIABLE *is a quantity which, during any set of mathematical operations, does not retain the same value but is capable of assuming different values.*

(*c*) *An* INDEPENDENT VARIABLE *is one which may take up any arbitrary value that may be assigned to it.*

(*d*) *A* DEPENDENT VARIABLE *is one which assumes its value in consequence of some second variable or system of variables taking up any set of arbitrary values that may be assigned to them.*

(*e*) *When one quantity depends upon another or upon a system of others in such a manner as to assume a definite value when a system of definite values is given to the others it is called a* FUNCTION *of those others.*

4. Notation. The usual notation to express that one variable y is a function of another x is

$$y = f(x) \text{ or } y = F(x) \text{ or } y = \phi(x).$$

Occasionally the brackets are dispensed with when no confusion can thereby arise. Thus fx may be sometimes written for $f(x)$. If u be an unknown function of several variables x, y, z, we may express the fact by the equation $u = f(x, y, z)$.

5. It has become conventional to use the letters $a, b, c \ldots \alpha, \beta, \gamma \ldots$ from the beginning of the alphabet to denote constants and to retain later letters, such as u, v, w, x, y, z and the Greek letters ξ, η, ζ for variables.

6. Limiting Values. The following illustrations will explain the meaning of the term "LIMITING VALUE":

(1) We say $\cdot\dot{6} = \frac{2}{3}$, by which we mean that by taking enough sixes we can make $\cdot 666\ldots$ *differ by as little as we please* from $\frac{2}{3}$.

(2) The limit of $\frac{2x+3}{x+1}$ when x is indefinitely diminished is 3. For the difference between $\frac{2x+3}{x+1}$ and 3 is $\frac{x}{x+1}$, and by diminishing x indefinitely this *difference can be made less than any assignable quantity however small.*

The expression can also be written $\dfrac{2+\frac{3}{x}}{1+\frac{1}{x}}$, which shews that if x be increased indefinitely it can be made to continually approach and to differ *by less than any assignable quantity* from 2, which is therefore its limit in that case.

It is useful to adopt the notation $Lt_{x=a}$ to denote the words "the Limit when $x = a$ of."

Thus $$Lt_{x=0}\frac{2x+3}{x+1} = 3\,;\quad Lt_{x=\infty}\frac{2x+3}{x+1} = 2.$$

(3) If an equilateral polygon be inscribed in any closed curve and the sides of the polygon be decreased indefinitely, and at the same time their number be increased indefinitely, the polygon continually approximates to the form of the curve, and ultimately *differs from it in area by less than any assignable magnitude*, and the curve is said to be the *limit* of the polygon inscribed in it.

7. We thus arrive at the following general definition:

DEF. *The* LIMIT *of a function for an assigned value of the independent variable is that value from which the function may be made to differ by less than any assignable quantity however small by making the independent variable approach sufficiently near its assigned value.*

8. Undetermined forms. When a function involves the independent variable in such a manner that for a certain assigned value of that variable its value *cannot be found by simply substituting* that value of the variable, the function is said to take an undetermined form.

One of the commonest cases occurring is that of a fraction whose numerator and denominator both vanish for the value of the variable referred to.

Let the student imagine a triangle whose sides are made of a material capable of shrinking indefinitely till they are smaller than any conceivable quantity. To fix the ideas suppose it to be originally a triangle whose sides are 3, 4 and 5 inches long, and suppose that the shrinkage is uniform. As the shrinkage proceeds the sides retain the same mutual ratio and may at any instant be written $3m$, $4m$, $5m$ and the angles remain unaltered. It thus appears that though each of these sides is ultimately immeasurably small, and to all practical purposes zero, they still retain the same mutual ratio $3 : 4 : 5$ which they had before the shrinkage began.

These considerations should convince the student *that the ultimate ratio of two vanishing quantities is not necessarily zero or unity.*

9. Consider the fraction $\frac{x^2 - a^2}{x - a}$; what is its value when $x = a$? Both numerator and denominator vanish when x is put $= a$. But it would be incorrect to assume that the fraction therefore takes the value unity. It is equally incorrect to suppose the value to be zero for the reason that its numerator is evanescent; or that it is infinite since its denominator is evanescent, as the beginner is often fallaciously led to believe. If we wish to evaluate this expression we must *never put x actually equal to a.* We may however put $x = a + h$ where h is anything other than zero.

Thus $$\frac{x^2 - a^2}{x - a} = 2a + h,$$

and it is now apparent that by making h indefinitely small (so that the value of x is made to approach indefinitely closely to its assigned value a) we may make the expression differ from $2a$ by less than any assignable

quantity. Therefore $2a$ is the limiting value of the given fraction.

10. Two functions of the same independent variable are said to be *ultimately equal* when as the independent variable approaches indefinitely near its assigned value the *limit of their ratio* is unity.

Thus $\quad Lt_{\theta=0}\dfrac{\sin\theta}{\theta}=1$ by trigonometry,

and therefore when an angle is indefinitely diminished its sine and its circular measure are ultimately equal.

EXAMPLES.

1. Find the limit when $x=0$ of $\dfrac{y}{x^2}$,

(*a*) when $y=mx$,
(*b*) when $y=x^2/a$,
(*c*) when $y=ax^2+b$.

2. Find $Lt\dfrac{ax+b}{bx+a}$, (i) when $x=0$, (ii) when $x=\infty$.

3. Find $Lt_{x=a}\dfrac{x^3-a^3}{x-a}$; $Lt_{x=a}\dfrac{x^4-a^4}{x-a}$; $Lt_{x=a}\dfrac{x^5-a^5}{x^2-a^2}$.

4. Find the limit of $\dfrac{ax+\dfrac{b}{x}}{cx+\dfrac{d}{x}}$, (i) when $x=0$, (ii) when $x=\infty$.

5. Find $Lt_{x=1}\dfrac{3x^2-4x+1}{x^2-4x+3}$.

6. The opposite angles of a cyclic quadrilateral are supplementary. What does this proposition become in the limit when two angular points coincide?

7. Evaluate the fraction $\dfrac{x^3-6x^2+11x-6}{x^{-3}-6x^{-2}+11x^{-1}-6}$ for the values $x=\infty$, 3, 2, 1, $\frac{1}{2}$, $\frac{1}{3}$, 0, $-\infty$.

8. Evaluate $Lt_{x=1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ and $Lt_{x=0}\dfrac{\sqrt{1+x}-1}{x}$.

11. Four Important Limits. The following limits are important:

(I) $Lt_{\theta=0}\dfrac{\sin\theta}{\theta}=1$; $Lt_{\theta=0}\cos\theta=1$,

(II) $Lt_{x=1}\dfrac{x^n-1}{x-1}=n$,

(III) $Lt_{x=\infty}\left(1+\dfrac{1}{x}\right)^x=e$, where e is the base of the Napierian logarithms,

(IV) $Lt_{x=0}\dfrac{a^x-1}{x}=\log_e a$.

12. (I) The limits (I) can be found in any standard text-book on Plane Trigonometry.

13. (II) To prove $Lt_{x=1}\dfrac{x^n-1}{x-1}=n$. Let $x=1+z$. Then when x approaches unity z approaches zero. Hence we can consider z to be less than 1, and we may therefore apply the Binomial to the expansion of $(1+z)^n$ whatever n may be.

Thus $$Lt_{x=1}\frac{x^n-1}{x-1}=Lt_{z=0}\frac{(1+z)^n-1}{z}$$

$$=Lt_{z=0}\frac{nz+\dfrac{n(n-1)}{1.2}z^2+\dots}{z}$$

$$=Lt_{z=0}\left\{n+\frac{n(n-1)}{1.2}z+\dots\right\}=n.$$

14. (III) To prove $Lt_{x=\infty}\left(1+\dfrac{1}{x}\right)^x=e$.

Let $$y=\left(1+\frac{1}{x}\right)^x,$$

then $$\log_e y=x\log_e\left(1+\frac{1}{x}\right).$$

Now x is about to become infinitely large, and therefore $\frac{1}{x}$ may be throughout regarded as *less than unity*, and we may expand by the Logarithmic Theorem.

Thus $$\log_e y = x\left\{\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \ldots\right\}$$

$$= 1 - \frac{1}{2x} + \frac{1}{3x^2} - \ldots$$

$$= 1 - \frac{1}{x} \times [\text{a } convergent\ series].$$

Thus when x becomes infinitely large

$$Lt \log_e y = 1,$$

and $$Lt\, y = e,$$

i.e. $$Lt_{x=\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

COR. $$Lt_{x=\infty}\left(1 + \frac{a}{x}\right)^x = Lt_{\frac{x}{a}=\infty}\left\{\left(1 + \frac{a}{x}\right)^{\frac{x}{a}}\right\}^a = e^a.$$

15. (IV) To prove $Lt_{x=0}\dfrac{a^x - 1}{x} = \log_e a$

Assume the expansion for a^x, viz.

$$a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots,$$

which is shewn in Algebra to be a *convergent series*.

Hence $$\frac{a^x - 1}{x} = \log_e a + x\frac{(\log_e a)^2}{2!} + \ldots$$

$$= \log_e a + x \times [\text{a } convergent\ series].$$

And the limit of the right-hand side, when x is indefinitely diminished, is clearly $\log_e a$.

16. Method of procedure. The rule for evaluating a function which takes the undetermined form $\frac{0}{0}$ when the independent variable x ultimately coincides with its assigned value a is as follows:—

Put $x = a + h$ and *expand both numerator and denominator* of the fraction. It will now become apparent that the reason why both numerator and denominator ultimately vanish is that some power of h is a common factor of each. This should now be *divided out.* Finally let h diminish indefinitely so that x becomes ultimately a, and the true limiting value of the function will be clear.

In the particular case in which x is to become *zero* the expansion of numerator and denominator in powers of x should be at once proceeded with without any preliminary substitution for x.

In the case in which x is to become infinite, put

$$x = \frac{1}{y},$$

so that when x becomes ∞, y becomes 0.

Several other undetermined forms occur, viz. $0 \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, ∞^0, 1^∞, but they may be made to depend upon the form $\frac{0}{0}$ by special artifices.

The method thus indicated will be best understood by examining the mode of solution of the following examples:—

Ex. 1. *Find* $\quad Lt_{x=1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}.$

This is of the form $\frac{0}{0}$ if we put $x=1$. Therefore we put $x=1+h$ and expand. We thus obtain

$$Lt_{x=1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = Lt_{h=0} \frac{(1+h)^7 - 2(1+h)^5 + 1}{(1+h)^3 - 3(1+h)^2 + 2}$$

$$= Lt_{h=0}\frac{(1+7h+21h^2+\ldots)-2(1+5h+10h^2+\ldots)+1}{(1+3h+3h^2+\ldots)-3(1+2h+h^2)+2}$$

$$= Lt_{h=0}\frac{-3h+h^2+\ldots}{-3h+\ldots}$$

$$= Lt_{h=0}\frac{-3+h+\ldots}{-3+\ldots}$$

$$= \frac{-3}{-3} = 1.$$

It will be seen from this example that in the process of expansion it is only necessary in general *to retain a few of the lowest powers of h.*

Ex. 2. *Find* $$Lt_{x=0}\frac{a^x-b^x}{x}.$$

Here numerator and denominator both vanish if x be put equal to 0. We therefore expand a^x and b^x by the exponential theorem. Hence

$$Lt_{x=0}\frac{a^x-b^x}{x}$$

$$= Lt_{x=0}\frac{\left\{1+x\log_e a+\frac{x^2}{2!}(\log_e a)^2+\ldots\right\}-\left\{1+x\log_e b+\frac{x^2}{2!}(\log_e b)^2+\ldots\right\}}{x}$$

$$= Lt_{x=0}\left\{\log_e a-\log_e b+\frac{x}{2!}(\overline{\log_e a}|^2-\overline{\log_e b}|^2)+\ldots\right\}$$

$$= \log_e a-\log_e b=\log_e\frac{a}{b}.$$

Ex. 3. *Find* $$Lt_{x=0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}.$$

Since $$\frac{\tan x}{x}=\frac{1}{\cos x}\cdot\frac{\sin x}{x}$$

we have $$Lt_{x=0}\frac{\tan x}{x}=1.$$

Hence the form assumed by $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ is an undetermined form 1^∞ when we put $x=0$.

Expand $\sin x$ and $\cos x$ in powers of x. This gives

$$Lt_{x=0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=Lt_{x=0}\left(\frac{x-\frac{x^3}{3!}+\ldots}{x-\frac{x^3}{2!}+\ldots}\right)^{\frac{1}{x^2}}.$$

$$= Lt_{x=0}\left(1+\frac{x^2}{3}+\text{higher powers of } x\right)^{\frac{1}{x^2}}$$

$$Lt_{x=0}\left\{1+\frac{x^2}{3}(1+\ldots)\right\}^{\frac{1}{x^2}}$$

$$= Lt_{x=0}\left(1+\frac{x^2 l}{3}\right)^{\frac{1}{x^2}},$$

where l is a series in ascending powers of x whose first term (and therefore whose limit when $x=0$) is unity. Hence

$$Lt_{x=0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = Lt_{x=0}\left\{\left(1+\frac{x^2 l}{3}\right)^{\frac{3}{lx^2}}\right\}^{\frac{l}{3}} = e^{\frac{1}{3}}, \text{ by Art. 14.}$$

Ex. 4. *Find* $\qquad Lt_{x=1}\, x^{\frac{1}{1-x}}$.

This expression is of the undetermined form 1^∞.

Put $\qquad 1-x=y,$

and therefore, if $x=1$, $\qquad y=0;$

therefore $\qquad$ Limit required $= Lt_{y=0}(1-y)^{\frac{1}{y}} = e^{-1}$ (Art. 14).

Ex. 5. $\qquad Lt_{x=\infty}\, x\left(a^{\frac{1}{x}}-1\right)$.

This is of the undetermined form $\infty \times 0$.

Put $\qquad x=\frac{1}{y},$

therefore, if $x=\infty$, $\qquad y=0,$

and $\qquad$ Limit required $= Lt_{y=0}\dfrac{a^y-1}{y} = \log_e a$ (Art. 15).

17. The following Algebraical and Trigonometrical series are added, as they are wanted for immediate use. They should be learnt thoroughly.

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots$$

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots\ldots$$

$$a^x = 1+x\log_e a+\frac{x^2(\log_e a)^2}{2!}+\frac{x^3(\log_e a)^3}{3!}+\ldots\ldots$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$$

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots$$

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots\dots$$

$$\frac{1}{2}\log_e\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\dots$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\dots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\dots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\dots$$

$$\cosh x\left[\text{which}\equiv\frac{e^x+e^{-x}}{2}\right]=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\dots$$

$$\sinh x\left[\text{which}\equiv\frac{e^x-e^{-x}}{2}\right]=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\dots$$

EXAMPLES.

Find the values of the following limits :

1. $Lt_{x=0}\dfrac{a^x-1}{b^x-1}$.
2. $Lt_{x=1}\dfrac{x^{\frac{3}{2}}-1}{x^{\frac{5}{2}}-1}$.
3. $Lt_{x=1}\dfrac{x^m-1}{x^n-1}$.
4. $Lt_{x=0}\dfrac{(1+x)^{\frac{1}{n}}-1}{x}$.
5. $Lt_{x=1}\dfrac{x^4+x^3-x^2-5x+4}{x^3-x^2-x+1}$.
6. $Lt_{x=1}\dfrac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1}$.
7. $Lt_{x=0}\dfrac{e^x-e^{-x}}{x}$.
8. $Lt_{x=0}\dfrac{e^x+e^{-x}-2}{x^2}$.
9. $Lt_{x=0}\dfrac{x\cos x-\log_e(1+x)}{x^2}$.
10. $Lt_{x=0}\dfrac{xe^x-\log_e(1+x)}{x^2}$.
11. $Lt_{x=0}\dfrac{x-\sin x\cos x}{x^3}$.
12. $Lt_{x=0}\dfrac{\sin^{-1}x-x}{x^3\cos x}$.
13. $Lt_{x=0}\dfrac{\cosh x-\cos x}{x\sin x}$.
14. $Lt_{x=0}\dfrac{\sin^{-1}x}{\tan^{-1}x}$.
15. $Lt_{x=0}\dfrac{\sin^{-1}x-\sinh x}{x^5}$.

16. $Lt_x \; x\cos^3 x - \log_e(1+x) - \sin^{-1}\frac{x^2}{2}$

$$\frac{2\sin x + \frac{1}{2}\log_e\frac{1+x}{1-x} - 3x}{}$$

$${}_{x=0}\;\frac{e^x\sin x - x - x^2}{x^2 + x\log_e(1-x)}.$$

19. $Lt_{x=0}\dfrac{x^3 e^{\frac{x^2}{4}} - \sin^{\frac{3}{2}} x^2}{x^7}.$

20. $Lt_{x=0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}.$

21. $Lt_{x=0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}.$

22. $Lt_{x=0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}.$

23. $Lt_{x=0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$

24. $Lt_{x=0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^3}}.$

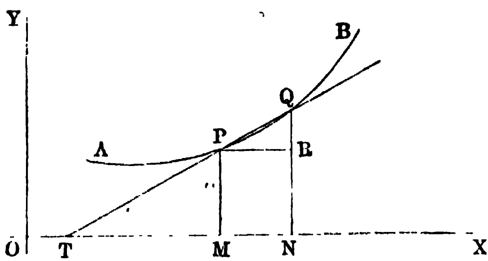
25. $Lt_{x=0}(\text{covers } x)^{\frac{1}{x}}.$

26. $Lt_{x=\frac{\pi}{2}}(\text{cosec } x)^{\tan^2 x}.$

CHAPTER II.

DIFFERENTIATION FROM THE DEFINITION.

18. Tangent of a Curve; Definition; Direction. Let AB be an arc of a curve traced in the plane of the paper, OX a fixed straight line in the same plane. Let



P, Q, be two points on the curve; PM, QN, perpendiculars on OX, and PR the perpendicular from P on QN. Join P, Q, and let QP be produced to cut OX at T.

When Q, *travelling along the curve*, approaches indefinitely near to P, the limiting position of chord QP is called the TANGENT at P. QR and PR both ultimately vanish, but the limit of their ratio *is in general finite*; for $Lt\,\dfrac{RQ}{PR} = Lt \tan RPQ = Lt \tan XTP =$ *tangent of the angle which the tangent at P to the curve makes with OX.*

If $y=\phi(x)$ be the equation of the curve and x, $x+h$ the abscissae of the points P, Q respectively; then $MP=\phi(x)$, $NQ=\phi(x+h)$, $RQ=\phi(x+h)-\phi(x)$ and $PR=h$.

Thus $$Lt\frac{RQ}{PR}=Lt_{h=0}\frac{\phi(x+h)-\phi(x)}{h}.$$

Hence, to draw the tangent at any point (x, y) on the curve $y=\phi(x)$, we must draw a line through that point, making with the axis of x an angle whose tangent is

$$Lt_{h=0}\frac{\phi(x+h)-\phi(x)}{h};$$

and if this limit be called m, the equation of the tangent at $P(x, y)$ will be

$$Y-y=m(X-x),$$

X, Y being the current co-ordinates of any point on the tangent; for the line represented by this equation goes through the point (x, y), and makes with the axis of x an angle whose tangent is m.

19. DEF.—DIFFERENTIAL COEFFICIENT.

Let $\phi(x)$ denote any function of x, and $\phi(x+h)$ the same function of $x+h$; then $Lt_{h=0}\dfrac{\phi(x+h)-\phi(x)}{h}$ is called the FIRST DERIVED FUNCTION *or* DIFFERENTIAL COEFFICIENT of $\phi(x)$ *with respect to x.*

The operation of finding this limit is called *differentiating* $\phi(x)$.

20. Geometrical meaning. The geometrical meaning of the above limit is indicated in the last article, where it is shewn to be *the tangent of the angle ψ which the tangent at any definite point (x, y) on the curve $y=\phi(x)$ makes with the axis of x.*

21. We can now find the differential coefficient of any proposed function by investigating the value of the

above limit; but it will be seen later on that, by means of certain rules to be established in Chap. III. and a knowledge of the differential coefficients of certain standard forms to be investigated in Chap. IV., we can always avoid the labour of an *ab initio* evaluation.

Ex. 1. Find from the definition the differential coefficient of $\frac{x^2}{a}$, where a is constant; and the equation of the tangent to the curve $ay=x^2$.

Here $$\phi(x)=\frac{x^2}{a},$$

$$\phi(x+h)=\frac{(x+h)^2}{a},$$

therefore $$Lt_{h=0}\frac{\phi(x+h)-\phi(x)}{h}=Lt_{h=0}\frac{(x+h)^2-x^2}{ha}$$

$$=Lt_{h=0}\frac{2xh+h^2}{ha}=Lt_{h=0}\frac{(2x+h)}{a}$$

$$=\frac{2x}{a}.$$

The geometrical interpretation of this result is that, if a tangent be drawn to the parabola $ay=x^2$ at the point (x, y), it will be inclined to the axis of x at the angle $\tan^{-1}\frac{2x}{a}$.

The equation of the tangent is therefore

$$Y-y=\frac{2x}{a}(X-x).$$

Ex. 2. Find from the definition the differential coefficient of $\log_e \sin\frac{x}{a}$, where a is a constant.

Here $$\phi(x)=\log_e \sin\frac{x}{a},$$

and
$$Lt_{h=0}\frac{\phi(x+h)-\phi(x)}{h}=Lt_{h=0}\frac{\log_e \sin\frac{x+h}{a}-\log_e \sin\frac{x}{a}}{h}$$

$$=Lt_{h=0}\frac{1}{h}\log_e\frac{\sin\frac{x}{a}\cos\frac{h}{a}+\cos\frac{x}{a}\sin\frac{h}{a}}{\sin\frac{x}{a}}$$

$$=Lt_{h=0}\frac{1}{h}\log_e\left(1+\frac{h}{a}\cot\frac{x}{a}+\text{higher powers of } h\right)$$

[by substituting for $\sin\frac{h}{a}$ and $\cos\frac{h}{a}$ their expansions in powers of $\frac{h}{a}$]

$$=Lt_{h=0}\frac{\frac{h}{a}\cot\frac{x}{a}-\text{higher powers of } h}{h}$$

[by expanding the logarithm]

$$=\frac{1}{a}\cot\frac{x}{a}.$$

Hence the tangent at any point on the curve $\frac{y}{a}=\log_e\sin\frac{x}{a}$ is inclined to the axis of x at an angle whose tangent is $\cot\frac{x}{a}$; that is at an angle $\frac{\pi}{2}-\frac{x}{a}$, and the equation of the tangent at the point x, y is

$$Y-y=\cot\frac{x}{a}(X-x)$$

EXAMPLES.

Find the equation of the tangent at the point (x, y) on each of the following curves

1. $y=x^3$.
2. $y=x^4$.
3. $y=\sqrt{x}$.
4. $y=x^2+x^3$
5. $y=\sin x$
6. $y=e^x$.
7. $y=\log_e x$.
8. $y=\tan x$
9. $x^2+y^2=c^2$.
10. $x^2/a^2+y^2/b^2=1$

22. Notation. It is convenient to use the notation δx for the same quantity which we have denoted by h, viz. a small but finite increase in the value of x. We may similarly denote by δy the *consequent* change in the value of y. Thus if (x, y), $(x+\delta x, y+\delta y)$ be contiguous points upon a given curve $y=\phi(x)$, we have

$$y+\delta y=\phi(x+\delta x),$$

and
$$\delta y=\phi(x+\delta x)-\phi(x).$$

Thus the differential coefficient

$$Lt_{\delta x=0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}$$

may be written

$$Lt_{\delta x=0}\frac{\delta y}{\delta x},$$

which more directly indicates the geometrical meaning

$$Lt_{PR=0}\frac{RQ}{PR}$$

pointed out in Art. 18.

The result of the operation expressed by

$$Lt_{h=0}\frac{\phi(x+h)-\phi(x)}{h},$$

or by $\quad Lt_{\delta x=0}\frac{\delta y}{\delta x},$

is denoted by $\quad \frac{d}{dx}y$ or $\frac{dy}{dx}$.

The student must *guard against the fallacious notion that dx and dy are separate small quantities*, as δx and δy are. He must remember that $\frac{d}{dx}$ is a *symbol of operation* which when applied to any function $\phi(x)$ means that we are

(1) *to increase x to $x+h$,*
(2) *to subtract the original value of the function,*
(3) *to divide the remainder by h,*
(4) *to evaluate the limit when h ultimately vanishes.*

Other notations expressing the same thing are

$$\frac{d\phi(x)}{dx},\ \frac{d\phi}{dx},\ \phi'(x),\ \phi',\ \dot{\phi},\ \phi_x,\ y',\ \dot{y},\ y_1.$$

EXAMPLES.

Find $\frac{dy}{dx}$ in the following cases.

1. $y=2x$. 2. $y=2+x$. 3. $y=2+3x$.

4. $y=2+3x^2$. **5.** $y=\frac{1}{x}$. **6.** $y=\frac{1}{x}+a$.

7. $y=\frac{1}{x^2}+a$. **8.** $y=a\sqrt{x}$. **9.** $y=\sqrt{x^2+a^2}$.

10. $y=e^{\sqrt{x}}$. **11.** $y=e^{\sin x}$. **12.** $y=\log_e \sec x$.

13. $y=x\sin x$. **14.** $y=\frac{\sin x}{x}$. **15.** $y=x^x$.

√ **23. Aspect of the Differential Coefficient as a Rate-Measurer.** When a particle is in motion in a given manner the space described is a function of the time of describing it. We may consider the time as an independent variable, and the space described in that time as the dependent variable.

The rate of change of position of the particle is called its velocity.

If *uniform* the velocity is measured by the space described in one second; if *variable*, the velocity at any instant is measured by the space which would be described in one second if, for that second, the velocity remained unchanged.

Suppose a space s to have been described in time t with varying velocity, and an additional space δs to be described in the additional time δt. Let v_1 and v_2 be the greatest and least values of the velocity during the interval δt; then the spaces which would have been described with uniform velocities v_1, v_2, in time δt are $v_1\delta t$ and $v_2\delta t$, and are respectively greater and less than the actual space δs.

Hence v_1, $\frac{\delta s}{\delta t}$, and v_2 are in descending order of magnitude.

If then δt be diminished indefinitely, we have in the limit $v_1 = v_2 =$ the velocity at the instant considered, which is therefore represented by $Lt\,\frac{\delta s}{\delta t}$, i.e. by $\frac{ds}{dt}$.

24. It appears therefore that we may give another interpretation to a differential coefficient, viz. that $\frac{ds}{dt}$ means the *rate of increase of s in point of time.* Similarly $\frac{dx}{dt}$, $\frac{dy}{dt}$, mean the *rates of change* of x and y respectively in point of time, and *measure the velocities*, resolved parallel to the axes, of a moving particle whose coordinates at the instant under consideration are x, y. If x and y be given functions of t, and therefore the path of the particle defined, and if δx, δy, δt, be simultaneous infinitesimal increments of x, y, t, then

$$\frac{dy}{dx} = Lt.\frac{\delta y}{\delta x} = Lt\frac{\dfrac{\delta y}{\delta t}}{\dfrac{\delta x}{\delta t}} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$$

and therefore represents *the ratio of the rate of change of y to that of x.* The rate of change of x is arbitrary, and if we choose it to be unit velocity, then

$$\frac{dy}{dx} = \frac{dy}{dt} = \text{absolute rate of change of } y.$$

25. Meaning of Sign of Differential Coefficient. If x be increasing with t, the x-velocity is positive, whilst, if x be decreasing while t increases, that velocity is negative. Similarly for y.

Moreover, since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{dy}{dx}$ is *positive when x and y increase or decrease together, but negative when one increases as the other decreases.*

This is obvious also from the geometrical interpretation of $\frac{dy}{dx}$. For, if x and y are *increasing together,*

$\frac{dy}{dx}$ is the *tangent of an acute angle and therefore positive*, while if, *as x increases y decreases*, $\frac{dy}{dx}$ *represents the tangent of an obtuse angle and is negative.*

26. The above article frequently affords important information with regard to the sign of a given expression. For if, for instance, $\phi(x)$ be a continuous function which is positive when $x=a$ and when $x=b$, and if $\phi'(x)$ be of one sign for all values of x lying between a and b so that it is known that $\phi(x)$ is always increasing or always decreasing from the one value $\phi(a)$ to the other $\phi(b)$, it will follow that $\phi(x)$ must be positive for all intermediate values of x.

Ex. Let $\phi(x)=(x-1)e^x+1.$

Here $\phi(0)=0$ and $\phi'(x)=Lt_{h=0}\frac{(x+h-1)e^{x+h}-(x-1)e^x}{h}$

$$=Lt_{h=0}\frac{(x+h-1)(1+h+\ldots)-(x-1)}{h}e^x$$

$$=Lt_{h=0}\frac{hx+\text{higher powers of } h}{h}e^x=xe^x.$$

So that $\phi'(x)$ is positive for all positive values of x. Therefore as x increases from 0 to ∞, $\phi(x)$ is always increasing. Hence since its initial value is zero the expression is positive for all positive values of x.

EXAMPLES.

1. Differentiate the following expressions, and shew that they are each positive for all positive values of x:

(i) $(x-2)e^x+x+2$,

(ii) $(x-3)e^x+\frac{x^2}{2}+2x+3$,

(iii) $x-\log_e(1+x)$.

2. In the curve $y=ce^{\frac{x}{c}}$, if ψ be the angle which the tangent at any point makes with the axis of x, prove $y=c\tan\psi$.

3. In the curve $y=c\cosh\frac{x}{c}$, prove $y=c\sec\psi$.

4. In the curve $3b^2y = x^3 - 3ax^2$ find the points at which the tangent is parallel to the axis of x.

[N.B.—This requires that $\tan\psi = 0$.]

5. Find at what points of the ellipse $x^2/a^2 + y^2/b^2 = 1$ the tangent cuts off equal intercepts from the axes.

[N.B.—This requires that $\tan\psi = \pm 1$.]

6. Prove that if a particle move so that the space described is proportional to the square of the time of description, the velocity will be proportional to the time, and the rate of increase of the velocity will be constant.

7. Shew that if a particle moves so that the space described is given by $s \propto \sin\mu t$, where μ is a constant, the rate of increase of the velocity is proportional to the distance of the particle measured along its path from a fixed position.

8. Shew that the function

$$x \sin x + \cos x + \cos^2 x$$

continually diminishes as x increases from 0 to $\pi/2$

9. If $$y = 2x - \tan^{-1} x - \log_e (x + \sqrt{1 + x^2}),$$
shew that y continually increases as x changes from zero to positive infinity.

10. A triangle has two of its angular points at $(a, 0)$, $(0, b)$, and the third (x, y) is moveable along the line $y = x$. Shew that if A be its area

$$2\frac{dA}{dx} = a + b,$$

and interpret this result geometrically.

11. If A be the area of a circle of radius x, shew that the circumference is $\dfrac{dA}{dx}$. Interpret this geometrically.

12. O is a given point and NP a given straight line upon which ON is the perpendicular. The radius OP rotates about O with the constant angular velocity ω. Shew that NP increases at the rate

$$\omega \, . \, ON \sec^2 NOP.$$

CHAPTER III.

FUNDAMENTAL PROPOSITIONS.

27. It will often be convenient in proving standard results to denote by a small letter the function of x considered, and by the corresponding capital the same function of $x+h$, e.g. if $u=\phi(x)$, then $U=\phi(x+h)$, or if $u=a^x$, then $U=a^{x+h}$.

Accordingly we shall have

$$\frac{du}{dx}=Lt_{h=0}\frac{U-u}{h},$$

$$\frac{dv}{dx}=Lt_{h=0}\frac{V-v}{h}.$$

etc.

We now proceed to the consideration of several important propositions.

28. Prop. I. **The Differential Coefficient of any Constant is zero.** This proposition will be obvious when we refer to the definition of a constant quantity. A constant is essentially a quantity of which there is no variation, so that if $y=c$, $\delta y=$ absolute zero whatever may be the value of δx. Hence $\frac{\delta y}{\delta x}=0$ and $\frac{dy}{dx}=0$ when the limit is taken.

Or geometrically: $y=c$ is the equation of a straight line parallel to the x-axis. At each point of its length it is its own tangent and makes an angle whose tangent is zero with the x-axis.

29. PROP. II. **Product of Constant and Function.** *The differential coefficient of a product of a constant and a function of x is equal to the product of the constant and the differential coefficient of the function*, or, stated algebraically,

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

For
$$\frac{d}{dx}(cu) = Lt_{h=0}\frac{cU - cu}{h} = cLt_{h=0}\frac{U-u}{h}$$
$$= c\frac{du}{dx}.$$

30. PROP. III. **Differential Coefficient of a Sum.** *The differential coefficient of the sum of a set of functions of x is the sum of the differential coefficients of the several functions.*

Let u, v, w, ..., be the functions of x, and y their sum.

Let U, V, W, ..., Y be what these expressions severally become when x is changed to $x+h$.

Then
$$y = u + v + w + \dots$$
$$Y = U + V + W + \dots,$$

and therefore
$$Y - y = (U-u) + (V-v) + (W-w) + \dots;$$

dividing by h,
$$\frac{Y-y}{h} = \frac{U-u}{h} + \frac{V-v}{h} + \frac{W-w}{h} + \dots$$

and taking the limit
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots.$$

If some of the connecting signs had been $-$ instead of $+$ a corresponding result would immediately follow, e.g. if
$$y = u + v - w + \dots,$$

then
$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}+\dots.$$

31. Prop. IV. **The Differential Coefficient of the product of two functions is**

$$(\textit{First Function}) \times (\textit{Diff. Coeff. of Second})$$
$$+ (\textit{Second Function}) \times (\textit{Diff. Coeff. of First}),$$

or, stated algebraically,

$$\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

With the same notation as before, let

$$y=uv, \text{ and therefore } Y=UV;$$

whence
$$Y-y=UV-uv$$
$$=u(V-v)+V(U-u);$$

therefore
$$\frac{Y-y}{h}=u\frac{V-v}{h}+V\frac{U-u}{h},$$

and taking the limit

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

32. On division by uv the above result may be written

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}.$$

Hence it is clear that the rule may be extended to products of more functions than two.

For example, if $y=uvw$; let $vw=z$, then $y=uz$.

Whence
$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{z}\frac{dz}{dx},$$

but
$$\frac{1}{z}\frac{dz}{dx}=\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx},$$

whence by substitution

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}.$$

Generally, if $\quad y=uvwt\ldots$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}+\frac{1}{t}\frac{dt}{dx}+\ldots,$$

and if we multiply by $uvwt\ldots$ we obtain

$$\frac{dy}{dx}=(vwt\ldots)\frac{du}{dx}+(uwt\ldots)\frac{dv}{dx}+(uvt\ldots)\frac{dw}{dx}+\ldots,$$

i.e. *multiply the differential coefficient of each separate function by the product of all the remaining functions and add up all the results;* the sum will be the differential coefficient of the product of all the functions.

33. PROP. V. **The Differential Coefficient of a quotient of two functions is**

$$\frac{(\textit{Diff. Coeff. of Num}^r.)\,(\textit{Den}^r.)-(\textit{Diff. Coeff. of Den}^r.)\,(\textit{Num}^r.)}{\textit{Square of Denominator}}$$

or, stated algebraically,

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\dfrac{du}{dx}v-\dfrac{dv}{dx}u}{v^2}.$$

With the same notation as before, let

$$y=\frac{u}{v},\ \text{and therefore}\ Y=\frac{U}{V},$$

whence $\quad Y-y=\dfrac{U}{V}-\dfrac{u}{v}$

$$=\frac{Uv-Vu}{Vv};$$

therefore $$\frac{Y-y}{h} = \frac{\dfrac{U-u}{h}v - \dfrac{V-v}{h}u}{Vv},$$

and taking the limit

$$\frac{dy}{dx} = \frac{\dfrac{du}{dx}v - \dfrac{dv}{dx}u}{v^2}.$$

34. To illustrate these rules let the student recall to memory the differential coefficients of x^2 and $a \log_e \sin \frac{x}{a}$ established in Art. 21, viz. $2x$ and $\cot \frac{x}{a}$ respectively.

Ex. 1. Thus if $\qquad y = x^2 + a \log_e \sin \frac{x}{a}$,

we have by Prop. III. $\qquad \frac{dy}{dx} = 2x + \cot \frac{x}{a}$.

Ex. 2. If $\qquad y = x^2 \times a \log_e \sin \frac{x}{a}$,

we have by Prop. IV. $\qquad \frac{dy}{dx} = 2x \times a \log_e \sin \frac{x}{a} + x^2 \cot \frac{x}{a}$.

Ex. 3. If $\qquad y = \dfrac{a \log_e \sin \frac{x}{a}}{x^2}$,

we have by Prop. V. $\qquad \frac{dy}{dx} = \dfrac{x^2 . \cot \frac{x}{a} - 2x . a \log_e \sin \frac{x}{a}}{x^4}$.

EXAMPLES.

[The following differential coefficients obtained as results of preceding examples may for *present purposes* be assumed :

$y = x^3, \quad y_1 = 3x^2. \qquad\qquad y = e^x \quad y_1 = e^x$

$y = x^4, \quad y_1 = 4x^3. \qquad\qquad y = \log_e x, \quad y_1 = \frac{1}{x}.$

$y = \sqrt{x}, \quad y_1 = \frac{1}{2\sqrt{x}}. \qquad\qquad y = \tan x, \quad y_1 = \sec^2 x.$

$y = \sin x, \quad y_1 = \cos x. \qquad\qquad y = \log_e \sin x, \quad y_1 = \cot x.$]

Differentiate the following expressions by aid of the foregoing rules :

1. $x^3 \sin x$, $x^3 e^x$, $x^3 \log_e x$, $x^3 \tan x$, $x^3 \log_e \sin x$.
2. $x^4/\sin x$, $\sin x/x^4$, $\sin x/e^x$, $e^x/\sin x$.
3. $\tan x \cdot \log_e \sin x$, $e^x \log_e x$, $\sin^2 x/\cos x$.
4. $x^3 e^x \sin x$, $x \tan x \log_e x$.
5. $x^3 \sin x/e^x$, $x^3/e^x \sin x$, $1/x^3 e^x \sin x$.
6. $2\sqrt{x} \cdot \sin x$, $3 \tan x/\sqrt{x}$, $5+4e^x/\sqrt{x}$.
7. $e^x(x^3+\sqrt{x})$, $(x^3+x^4)(e^x \log_e x)$.

35. Function of a function.

Suppose $u=f(v)$ ....(1),

where $v=\phi(x)$........................(2).

If x be changed to $x+\delta x$, v will become $v+\delta v$, and in consequence u will become $u+\delta u$.

Now if v had been first eliminated between equations (1) and (2) we should have a result of the form

$u=F(x)$ ....(3).

This equation will be satisfied by the same simultaneous values $x+\delta x$, $u+\delta u$, which satisfy equations (1) and (2). Also

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta v} \cdot \frac{\delta v}{\delta x};$$

and proceeding to the limit

$$Lt_{\delta x=0}\frac{\delta u}{\delta x}=\frac{du}{dx} \text{ as obtained from equation (3),}$$

$$Lt_{\delta v=0}\frac{\delta u}{\delta v}=\frac{du}{dv} \text{ as obtained from equation (1),}$$

$$Lt_{\delta x=0}\frac{\delta v}{\delta x}=\frac{dv}{dx} \text{ as obtained from equation (2).}$$

Thus $$\frac{du}{dx}=\frac{du}{dv} \cdot \frac{dv}{dx}.$$

36. For instance, the diff. coeff. of x^2 is $2x$, and of $\log_e \sin x$ is $\cot x$. } Art. 21.

Suppose $u = (\log_e \sin x)^2$, i.e. v^2 where $v = \log_e \sin x$, then

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = 2v \cdot \cot x = 2 \cot x \cdot \log_e \sin x.$$

37. It is obvious that the above result may be extended. For, if $u = \phi(v)$, $v = \psi(w)$, $w = f(x)$, we have

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx},$$

but

$$\frac{dv}{dx} = \frac{dv}{dw} \cdot \frac{dw}{dx};$$

and therefore

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx},$$

and a similar result holds however many functions there may be.

The rule may be expressed thus:

$$\frac{d\,(\text{1st Func.})}{dx} = \frac{d\,(\text{1st Func.})}{d\,(\text{2nd Func.})} \cdot \frac{d\,(\text{2nd Func.})}{d\,(\text{3rd Func.})} \cdots \frac{d\,(\text{Last Func.})}{dx}$$

or if $u = \phi\,[\psi\,\{F(fx)\}]$,

$$\frac{du}{dx} = \phi'[\psi\{F(fx)\}] \times \psi'\,\{F(fx)\} \times F'(fx) \times f'x.$$

Thus in the preceding Example

$$\frac{d\,(\log_e \sin x)^2}{dx} = \frac{d\,(\log_e \sin x)^2}{d\,(\log_e \sin x)} \cdot \frac{d \log_e \sin x}{dx} = 2 \log_e \sin x \cdot \cot x.$$

Again,

$$\frac{d\,(\log_e \sin x^2)}{dx} = \frac{d\,(\log_e \sin x^2)}{d\,(\sin x^2)} \cdot \frac{d \sin x^2}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \frac{1}{\sin x^2} \cdot \cos x^2 \cdot 2x = 2x \cot x^2.$$

38. Interchange of the dependent and independent variable. If in the theorem

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}$$

we put $$u=x,$$

then $$\frac{du}{dx}=\frac{dx}{dx}=Lt_{h=0}\frac{(x+h)-x}{h}=1,$$

and we obtain the result

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1,$$

or $$\frac{dy}{dx}=\frac{1}{\dfrac{dx}{dy}}.$$

39. The truth of this is also manifest geometrically, for $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ are respectively the tangent and the co-tangent of the angle ψ which the tangent to the curve $y=f(x)$ makes with the x-axis.

40. This formula is very useful in the differentiation of an inverse function.

Thus if we have $$y=f^{-1}(x),$$

$$x=f(y),$$

and $$\frac{dx}{dy}=f'(y),$$

a form which we are supposing to have been investigated.

Thus $$\frac{dy}{dx}=\frac{1}{f'(y)}=\frac{1}{f'[f^{-1}(x)]}.$$

EXAMPLES.

Assuming as before for *present purposes* the following differential coefficients,

$$\frac{d}{dx}x^3=3x^2, \quad \frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}, \quad \frac{d}{dx}\sin x=\cos x,$$

$$\frac{d}{dx}e^x=e^x, \quad \frac{d}{dx}\log_e x=\frac{1}{x}, \quad \frac{d}{dx}\tan x=\sec^2 x,$$

write down the differential coefficients of the following combinations:

1. e^{3x}, e^{-x}, $\sin^3 x$, $\sqrt{\sin x}$, $\sqrt{\log_e x}$, $\sqrt{\tan x}$, $\sin\sqrt{x}$.
2. $e^{\sin x}$, $e^{\tan x}$, e^{x^3}, $e^{\sqrt{x}}$, $e^{\log_e x}$.
3. $\log_e \sin x$, $\log_e \tan x$, $\log_e \sqrt{x}$, $\log_e x^{-1}$.
4. $\sin\log_e x$, $\tan\log_e x$, $\sqrt{\sin\log_e x}$, $\sqrt{\sin\sqrt{x}}$, $\log_e \sin\sqrt{x}$.
5. $\log_e\sqrt{\sin\sqrt{e^x}}$, $\tan\log_e \sin e^{\sqrt{x}}$.

CHAPTER IV.

STANDARD FORMS.

41. It is the object of the present Chapter to investigate and tabulate the results of differentiating the several standard forms referred to in Art. 21.

We shall always consider angles to be measured in circular measure, and all logarithms to be Napierian, unless the contrary is expressly stated.

It will be remembered that if $u = \phi(x)$, then, by the definition of a differential coefficient,

$$\frac{du}{dx} = Lt_{h=0} \frac{\phi(x+h) - \phi(x)}{h}.$$

42. Differential Coefficient of x^n.

If $\quad u = \phi(x) = x^n,$

then $\quad \phi(x+h) = (x+h)^n,$

and $\quad \dfrac{du}{dx} = Lt_{h=0} \dfrac{(x+h)^n - x^n}{h}$

$$= Lt_{h=0}\, x^n \frac{\left(1 + \frac{h}{x}\right)^n - 1}{h}.$$

…, since h is to be ultimately zero, we may consider … be less than unity, and we can therefore apply the

Binomial Theorem to expand $\left(1+\frac{h}{x}\right)^n$, whatever be the value of n; hence

$$\frac{du}{dx}=Lt_{h=0}\frac{x^n}{h}\left\{n\frac{h}{x}+\frac{n(n-1)}{2!}\frac{h^2}{x^2}+\frac{n(n-1)(n-2)}{3!}\frac{h^3}{x^3}+\dots\right\}$$

$$=Lt_{h=0}\,nx^{n-1}\left\{1+\frac{h}{x}\times(\text{a } \textit{convergent series})\right\}$$

$$=nx^{n-1}.$$

43. It follows by Art. 35 that if $u=[\phi(x)]^n$ then

$$\frac{du}{dx}=n\,[\phi(x)]^{n-1}\,\phi'(x).$$

EXAMPLES.

Write down the differential coefficients of

1. x, x^{10}, x^{-1}, x^{-10}, $x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{-\frac{5}{6}}$, $\sqrt{x^{-5}}$.

2. $(x+a)^n$, x^n+a^n, $x^{\frac{1}{2}}+a^{\frac{1}{2}}$, $\frac{1}{x+a}$, $\frac{1}{\sqrt{x+a}}$.

3. $(ax+b)^n$, ax^n+b, $(ax)^n+b$, $a(x+b)^n$, $a^n(x+b)$.

4. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\dots\dots$

5. $\frac{a+b\sqrt[3]{x}}{c\sqrt[4]{x^5}}$, $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$, $\frac{\sqrt[3]{a}+\sqrt[3]{x}}{\sqrt[3]{a}-\sqrt[3]{x}}$, $\sqrt{\frac{a+x}{a-x}}$, $\sqrt[3]{\frac{a+x}{a-x}}$.

6. $\frac{ax^2+bx+c}{cx^2+bx+a}$, $(x+a)^p(x+b)^q$, $(x+a)^p/(x+b)^q$.

44. Differential Coefficient of a^x.

If $$u=\phi(x)=a^x,$$

$$\phi(x+h)=a^{x+h},$$

and
$$\frac{du}{dx}=Lt_{h=0}\frac{a^{x+h}-a^x}{h}$$
$$=a^x Lt_{h=0}\frac{a^h-1}{h}$$
$$=a^x \log_e a. \quad \text{[Art. 15.]}$$

COR. 1. If $u=e^x$, $\dfrac{du}{dx}=e^x\log_e e=e^x$.

COR. 2. It follows by Art. 35 that if $u=e^{\phi(x)}$, then
$$\frac{du}{dx}=e^{\phi(x)}\,.\,\phi'(x).$$

45. Differential Coefficient of $\log_a x$.

If
$$u=\phi(x)=\log_a x,$$
$$\phi(x+h)=\log_a(x+h),$$
and
$$\frac{du}{dx}=Lt_{h=0}\frac{\log_a(x+h)-\log_a x}{h}$$
$$=Lt_{h=0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right).$$

Let $\dfrac{x}{h}=z$, so that if $h=0$, $z=\infty$; therefore
$$\frac{du}{dx}=Lt_{z=\infty}\frac{z}{x}\log_a\left(1+\frac{1}{z}\right)$$
$$=\frac{1}{x}Lt_{z=\infty}\log_a\left(1+\frac{1}{z}\right)^z$$
$$=\frac{1}{x}\log_a e. \quad \text{[Art. 14.]}$$

COR. 1. If $u=\log_e x$, $\dfrac{du}{dx}=\dfrac{1}{x}\log_e e=\dfrac{1}{x}$.

COR. 2. And it follows as before that if
$$u=\log_e\phi(x),$$
then
$$\frac{du}{dx}=\frac{\phi'(x)}{\phi(x)}.$$

EXAMPLES.

Write down the differential coefficients of

1. e^{2x}, e^{-x}, e^{nx}, $\cosh x$, $\sinh x$, $\dfrac{e^{2x}+e^{3x}}{1+e^{-x}}$.

2. $\log\sqrt{x}$, $\log(x+a)$, $\log(ax+b)$, $\log(ax^2+bx+c)$,
$\log\dfrac{1+x}{1-x}$, $\log\dfrac{1+x^2}{1-x^2}$, $\log_x a$.

3. $\phi(e^x)$, $\phi(\log x)$, $[\phi(x)]^{\frac{1}{2}}$, $[\phi(a+x)]^n$, $\phi[(a+x)^n]$.

4. $e^x\log(x+a)$, $x^n e^x$, $a^x . e^x$, 2^x, x° (degrees).

5. $\log(x+e^x)$, $e^x+\log x$, $e^x/\log x$.

6. $e^{x\log x}$, $\log(xe^x)$, $\log x^x$.

46. Differential Coefficient of $\sin x$.

If
$$u=\phi(x)=\sin x,$$
$$\phi(x+h)=\sin(x+h),$$
and
$$\frac{du}{dx}=Lt_{h=0}\frac{\sin(x+h)-\sin x}{h}$$
$$=Lt_{h=0}\frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h}$$
$$=Lt_{h=0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cos\left(x+\frac{h}{2}\right)$$
$$=\cos x. \quad \text{[Art. 11; I.]}$$

47. Differential Coefficient of $\cos x$.

If
$$u=\phi(x)=\cos x,$$
$$\phi(x+h)=\cos(x+h),$$
and
$$\frac{du}{dx}=Lt_{h=0}\frac{\cos(x+h)-\cos x}{h}$$

$$= -Lt_{h \to 0} \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} \sin\left(x + \frac{h}{2}\right)$$

$$= -\sin x.$$

COR. And as in previous cases the differential coefficients of $\sin \phi(x)$ and $\cos \phi(x)$ are respectively

$$\cos \phi(x) \,.\, \phi'(x),$$

and $$-\sin \phi(x) \,.\, \phi'(x).$$

EXAMPLES.

Write down the differential coefficients of

1. $\sin 2x$, $\sin nx$, $\sin^n x$, $\sin x^n$, $\sin \sqrt{x}$.
2. $\sqrt{\sin\sqrt{x}}$, $\log \sin x$, $\log \sin \sqrt{x}$, $e^{\sin x}$, $e^{\sqrt{\sin x}}$.
3. $\sin^m x \cos^n x$, $\sin^m x/\cos^n x$, $\sin^n (nx^n)$, $e^{ax} \sin bx$.
4. $\sin x \sin 2x \sin 3x$, $\sin x \,.\, \sin 2x/\sin 3x$.
5. $\cos x \cos 2x \cos 3x$, $\cos^p ax \,.\, \cos^q bx \,.\, \cos^r cx$.

48. The remaining circular functions can be differentiated from the definition in the same way. It is a little quicker however to proceed thus after obtaining the above results.

(i) If $y = \tan x = \dfrac{\sin x}{\cos x}$;

$$\frac{dy}{dx} = \frac{\dfrac{d}{dx}(\sin x) \,.\, \cos x - \dfrac{d}{dx}(\cos x) \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

(ii) If $y = \cot x = \dfrac{\cos x}{\sin x}$,

$$\frac{dy}{dx} = \frac{(-\sin x)\sin x - \cos x\,(\cos x)}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

(iii) If $y = \sec x = (\cos x)^{-1}$,

$$\frac{dy}{dx} = (-1)(\cos x)^{-2}\frac{d}{dx}(\cos x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

(iv) If $y = \operatorname{cosec} x = (\sin x)^{-1}$,

$$\frac{dy}{dx} = (-1)(\sin x)^{-2}\frac{d}{dx}(\sin x) = -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

(v) If $y = \operatorname{vers} x = 1 - \cos x$,

$$\frac{dy}{dx} = \sin x.$$

(vi) If $y = \operatorname{covers} x = 1 - \sin x$,

$$\frac{dy}{dx} = -\cos x.$$

49. Differentiation of the inverse functions.

We may deduce the differential coefficients of all the inverse functions directly from the definition as shewn below.

For this method it seems useful to recur to the notation of Art. 27 and to denote $\phi(x+h)$ by U.

50. Then if $u = \phi(x) = \sin^{-1} x$,

$$U = \phi(x+h) = \sin^{-1}(x+h).$$

Hence $\qquad x = \sin u$, and $x + h = \sin U$;

therefore $\qquad h = \sin U - \sin u$,

and $$\frac{du}{dx} = Lt_{h=0}\frac{U-u}{h} = Lt_{U=u}\frac{U-u}{\sin U - \sin u}$$

$$= Lt_{U=u}\left\{\frac{\frac{U-u}{2}}{\sin\frac{U-u}{2}}\right\}\frac{1}{\cos\frac{U+u}{2}}$$

$$=\frac{1}{\cos u}=\frac{1}{\sqrt{1-\sin^2 u}}=\frac{1}{\sqrt{1-x^2}},$$

and the remaining inverse functions may be differentiated similarly.

51. But the method indicated in the preceding chapter (Art. 40) for inverse functions simplifies and shortens the work considerably.

Thus:—

(i) If $\qquad u=\sin^{-1}x,$

we have $\qquad x=\sin u;$

whence $\qquad \frac{dx}{du}=\cos u;$

and therefore $\qquad \frac{du}{dx}=\frac{1}{\frac{dx}{du}}=\frac{1}{\cos u}=\frac{1}{\sqrt{1-\sin^2 u}}=\frac{1}{\sqrt{1-x^2}},$

and since $\quad \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x,$

we have $\quad \frac{d\cos^{-1}x}{dx}=-\frac{1}{\sqrt{1-x^2}}.$

(ii) If $\qquad u=\tan^{-1}x,$

we have $\qquad x=\tan u;$

whence $\qquad \frac{dx}{du}=\sec^2 u;$

and therefore $\qquad \frac{du}{dx}=\frac{1}{\sec^2 u}=\frac{1}{1+\tan^2 u}=\frac{1}{1+x^2},$

and since $\quad \cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x,$

we have $$\frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2}.$$

(iii) If $$u = \sec^{-1} x,$$
we have $$x = \sec u;$$
whence $$\frac{dx}{du} = \sec u \tan u;$$
and therefore $$\frac{du}{dx} = \frac{\cos^2 u}{\sin u} = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x\sqrt{x^2-1}};$$

and since $$\operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x:$$

we have $$\frac{d\,(\operatorname{cosec}^{-1} x)}{dx} = -\frac{1}{x\sqrt{x^2-1}}.$$

(iv) If $$u = \operatorname{vers}^{-1} x,$$
we have $$x = \operatorname{vers} u = 1 - \cos u;$$
whence $$\frac{dx}{du} = \sin u:$$
and therefore $$\frac{du}{dx} = \frac{1}{\sin u} = \frac{1}{\sqrt{1-\cos^2 u}} = \frac{1}{\sqrt{2x-x^2}};$$
whence also $$\frac{d\operatorname{covers}^{-1} x}{dx} = -\frac{1}{\sqrt{2x-x^2}}.$$

EXAMPLES.

Write down the differential coefficients of each of the following expressions :

1. $\sec x^2$, $\sec^{-1} x^2$, $\tan x^2$, $\tan^{-1} x^2$, $\operatorname{vers} x^2$, $\operatorname{vers}^{-1} x^2$.
2. $\tan^{-1} e^x$, $\tan e^x$, $\log \tan x$, $\log \tan^{-1} x$, $\log (\tan x)^{-1}$.
3. $\operatorname{vers}^{-1} \frac{x}{a}$, $\operatorname{vers}^{-1}(x+a)$, $\tan^{-1} \frac{x}{a}$, $\cos^{-1} \frac{1-x^2}{1+x^2}$.
4. $\sqrt{\operatorname{covers} x}$, $\tan^p x^q$, $(\tan^{-1} x^p)^q$, $x \log \tan^{-1} x$.
5. $\tan x \,.\, \sin^{-1} x$, $\sec^{-1} \tan x$, $\tan^{-1} \sec x$, $e^x \sin^{-1} x$.

52. Table of Results to be committed to memory.

$$u = x^n. \qquad \frac{du}{dx} = nx^{n-1}.$$

$$u = a^x. \qquad \frac{du}{dx} = a^x \log_e a.$$

$$u = e^x. \qquad \frac{du}{dx} = e^x.$$

$$u = \log_a x. \qquad \frac{du}{dx} = \frac{1}{x} \log_a e.$$

$$u = \log_e x. \qquad \frac{du}{dx} = \frac{1}{x}.$$

$$u = \sin x. \qquad \frac{du}{dx} = \cos x.$$

$$u = \cos x. \qquad \frac{du}{dx} = -\sin x.$$

$$u = \tan x. \qquad \frac{du}{dx} = \sec^2 x.$$

$$u = \cot x. \qquad \frac{du}{dx} = -\operatorname{cosec}^2 x.$$

$$u = \sec x. \qquad \frac{du}{dx} = \frac{\sin x}{\cos^2 x}.$$

$$u = \operatorname{cosec} x. \qquad \frac{du}{dx} = -\frac{\cos x}{\sin^2 x}.$$

$$u = \sin^{-1} x. \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$u = \cos^{-1} x. \qquad \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

$$u = \tan^{-1} x. \qquad \frac{du}{dx} = \frac{1}{1+x^2}.$$

$$u = \cot^{-1} x. \qquad \frac{du}{dx} = -\frac{1}{1+x^2}.$$

$$u = \sec^{-1}x. \qquad \frac{du}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

$$u = \operatorname{cosec}^{-1}x. \qquad \frac{du}{dx} = -\frac{1}{x\sqrt{x^2-1}}.$$

$$u = \operatorname{vers}^{-1}x. \qquad \frac{du}{dx} = \frac{1}{\sqrt{2x-x^2}}.$$

$$u = \operatorname{covers}^{-1}x. \qquad \frac{du}{dx} = -\frac{1}{\sqrt{2x-x^2}}.$$

53. The Form u^v. Logarithmic Differentiation.

In functions of the form u^v, where both u and v are functions of x, it is generally advisable to *take logarithms* before proceeding to differentiate.

Let $\qquad y = u^v,$

then $\qquad \log_e y = v\log_e u;$

therefore $\qquad \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}\cdot\log_e u + v\cdot\frac{1}{u}\frac{du}{dx}$, Arts. 31, 45,

or $$\frac{dy}{dx} = u^v\left(\log_e u\cdot\frac{dv}{dx} + \frac{v}{u}\frac{du}{dx}\right).$$

Three cases of this proposition present themselves.

1. If *v be a constant* and u a function of x, $\frac{dv}{dx} = 0$ and the above reduces to

$$\frac{dy}{dx} = v\cdot u^{v-1}\frac{du}{dx},$$

as might be expected from Art. 43.

II. If *u be a constant* and v a function of x, $\frac{du}{dx}=0$ and the general form proved above reduces to

$$\frac{dy}{dx}=u^v\log_e u\,.\,\frac{dv}{dx},$$

as might be expected from Art. 44.

III. If u and v be *both functions of* x, it appears that the general formula

$$\frac{dy}{dx}=u^v\log_e u\frac{dv}{dx}+vu^{v-1}\frac{du}{dx}$$

is the sum of the two special forms in I. and II., and therefore we may, instead of taking logarithms in any particular example, *consider first u constant and then v constant and add the results obtained on these suppositions.*

Ex. 1. Thus if $y=(\sin x)^x$,

$$\log y=x\log\sin x;$$

therefore $$\frac{1}{y}\frac{dy}{dx}=\log\sin x+x\cot x,$$

and $$\frac{dy}{dx}=(\sin x)^x\{\log\sin x+x\cot x\}.$$

Ex. 2. In cases such as $y=x^x+(\sin x)^x$, we cannot take logarithms directly. Let $u=x^x$ and $v=(\sin x)^x$.

Then $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

But $$\log u=x\log x,$$

and $$\log v=x\log\sin x;$$

whence $$\frac{du}{dx}=x^x\{1+\log x\},$$

and $$\frac{dv}{dx}=(\sin x)^x\{\log\sin x+x\cot x\},$$

and $\therefore$ $$\frac{dy}{dx}=x^x\{1+\log x\}+(\sin x)^x\{\log\sin x+x\cot x\}.$$

The above compound process is called **Logarithmic differentiation** and is useful whenever variables occur

as an index or when the expression to be differentiated consists of a product of several involved factors.

EXAMPLES.

1. Differentiate $x^{\sin x}$, $(\sin^{-1} x)^x$, x^{x^2}, x^{2x}.
2. Differentiate $(\sin x)^{\cos x}+(\cos x)^{\sin x}$, $(\tan x)^x+x^{\tan x}$.
3. Differentiate $\tan x\times\log x\times e^x\times x^x\times\sqrt{x}$.

54. Transformations. Occasionally an Algebraic or Trigonometrical transformation before beginning to differentiate will much shorten the work.

(i) For instance, suppose

$$y=\tan^{-1}\frac{2x}{1-x^2}.$$

We observe that $y=2\tan^{-1}x$;

whence $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$.

(ii) Suppose $y=\tan^{-1}\dfrac{1+x}{1-x}$.

Here $y=\tan^{-1}x+\tan^{-1}1$,

and therefore $\dfrac{dy}{dx}=\dfrac{1}{1+x^2}$.

(iii) If $y=\tan^{-1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

we have $y=\tan^{-1}\dfrac{1-\sqrt{\dfrac{1-x^2}{1+x^2}}}{1+\sqrt{\dfrac{1-x^2}{1+x^2}}}=\dfrac{\pi}{4}-\tan^{-1}\sqrt{\dfrac{1-x^2}{1+x^2}}=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x^2$;

$$\therefore \frac{dy}{dx}=\frac{x}{\sqrt{1-x^4}}.$$

EXAMPLES.

Differentiate :

1. $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$. 2. $\tan^{-1}\dfrac{p-qx}{q+px}$. 3. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$.
4. $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$. 5. $e^{\log x}$. 6. $\sec^{-1}\dfrac{1}{1-2x^2}$.

7. $\sec\tan^{-1}x$. **8.** $\cos^{-1}\dfrac{x-x^{-1}}{x+x^{-1}}$. **9.** $\sin^{-1}(3x-4x^3)$.

10. $\tan^{-1}\dfrac{\sqrt{x}-x}{1+x^{\frac{3}{2}}}$. **11.** $\cos^{-1}(1-2x^2)$. **12.** $\log\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$.

55. Examples of Differentiation.

Ex. 1. Let $y=\sqrt{z}$, where z is a known function of x.

Here $$y=z^{\frac{1}{2}},$$

and $$\frac{dy}{dz}=\tfrac{1}{2}z^{-\frac{1}{2}}=\frac{1}{2\sqrt{z}};$$

whence $$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}, \text{ (Art. 35)}$$
$$=\frac{1}{2\sqrt{z}}\cdot\frac{dz}{dx}.$$

This form *occurs so often that it will be found convenient to commit it to memory.*

Ex. 2. Let $$y=e^{\sqrt{\cot x}}.$$

Here $$\frac{d\left(e^{\sqrt{\cot x}}\right)}{dx}=\frac{d\left(e^{\sqrt{\cot x}}\right)}{d\left(\sqrt{\cot x}\right)}\cdot\frac{d\left(\sqrt{\cot x}\right)}{d(\cot x)}\cdot\frac{d(\cot x)}{dx}$$
$$=e^{\sqrt{\cot x}}\cdot\frac{1}{2\sqrt{\cot x}}\cdot(-\operatorname{cosec}^2 x).$$

Ex. 3. Let $$y=(\sin x)^{\log x}\cot\{e^x(a+bx)\}.$$

Taking logarithms
$$\log y=\log x\,.\log\sin x+\log\cot\{e^x(a+bx)\}.$$

The differential coefficient of $\log y$ is $\dfrac{1}{y}\dfrac{dy}{dx}$.

Again, $\log x\,.\log\sin x$ is a product, and when differentiated becomes (Art. 31)
$$\frac{1}{x}\log\sin x+\log x\,.\frac{1}{\sin x}\,.\cos x.$$

Also, $\log\cot\{e^x(a+bx)\}$ becomes when differentiated
$$\frac{1}{\cot\{e^x(a+bx)\}}\,.\left[-\operatorname{cosec}^2\{e^x(a+bx)\}\right].\{e^x(a+bx)+be^x\};$$

$$\therefore\quad\frac{dy}{dx}=(\sin x)^{\log x}.\cot\{e^x(a+bx)\}\left[\frac{1}{x}\log\sin x+\cot x\,.\log x\right.$$
$$\left.-2e^x(a+b+bx)\operatorname{cosec}2(e^xa+bx)\right].$$

Ex. 4. Let $y=\sqrt{a^2-b^2\cos^2(\log x)}$. Then

$$\frac{dy}{dx}=\frac{d\sqrt{a^2-b^2\cos^2(\log x)}}{d\{a^2-b^2\cos^2(\log x)\}}\times\frac{d\{a^2-b^2\cos^2(\log x)\}}{d\{\cos(\log x)\}}\times\frac{d\{\cos(\log x)\}}{d(\log x)}\times\frac{d(\log x)}{dx}$$

$$=\tfrac{1}{2}\{a^2-b^2\cos^2(\log x)\}^{-\frac{1}{2}}\times\{-2b^2\cos(\log x)\}\times\{-\sin(\log x)\}\times\frac{1}{x}$$

$$=\frac{b^2\sin 2(\log x)}{2x\sqrt{a^2-b^2\cos^2(\log x)}}.$$

Ex. 5. **Differentiate x^5 with regard to x^2.**

Let $$x^2=z.$$

Then $$\frac{dx^5}{dz}=\frac{dx^5}{dx}\cdot\frac{dx}{dz}=\frac{\dfrac{dx^5}{dx}}{\dfrac{dz}{dx}}=\frac{5x^4}{2x}$$

5

56. Implicit relation of x and y. So far we have been concerned with the case in which y is expressed *explicitly*, i.e. directly in terms of x.

Cases however are of frequent occurrence in which y is not expressed directly in terms of x, but its functionality is implied by an algebraic relation connecting x and y.

In the case of such an *implicit* relation we proceed as follows:—

Suppose for instance

$$x^3+y^3=3axy,$$

then $$3x^2+3y^2\frac{dy}{dx}=3a\left(y+x\frac{dy}{dx}\right),$$

i.e. $$3(x^2-ay)+3(y^2-ax)\frac{dy}{dx}=0,$$

giving $$\frac{dy}{dx}=\frac{ay}{y^2-ax}.$$

57. Partial Differentiation. It will be perceived in the foregoing example that the expressions $3(x^2 - ay)$ and $3(y^2 - ax)$ occurring are algebraically the same as would be given by differentiating the expression $x^3 + y^3 - 3axy$ first with regard to x, keeping y a constant, and second with regard to y, keeping x a constant.

When such processes are applied to a function $f(x, y)$ of two or more variables the results are denoted by the symbols $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Thus in the above example

$$\frac{\partial f}{\partial x} = 3(x^2 - ay),$$

and
$$\frac{\partial f}{\partial y} = 3(y^2 - ax).$$

This is termed *partial* differentiation, and the results are called partial differential coefficients.

58. A general proposition. It appears that in the preceding example

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

This proposition is true for all implicit relations between two variables, such as $f(x, y) = 0$.

Suppose the function capable of expansion by any means in powers of x and y, so that any general term may be denoted by Ax^py^q.

Then $$f(x, y) \equiv \Sigma A x^p y^q = 0.$$

Then differentiating

$$\Sigma\left(Apx^{p-1}y^q + Ax^p qy^{q-1}\frac{dy}{dx}\right) = 0,$$

or
$$\Sigma Apx^{p-1}y^q + (\Sigma Aqx^p y^{q-1})\frac{dy}{dx} = 0,$$

or
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0.$$

Ex. If
$$f(x, y) \equiv x^5 + x^4 y + y^3 = 0,$$

we have
$$\frac{\partial f}{\partial x} = 5x^4 + 4x^3 y,$$

$$\frac{\partial f}{\partial y} = x^4 + 3y^2;$$

$$\therefore \frac{dy}{dx} = -\frac{5x^4 + 4x^3 y}{x^4 + 3y^2}.$$

EXAMPLES.

Find $\frac{dy}{dx}$ in the following cases :

1. $x^3 + y^3 = a^3$.
2. $x^n + y^n = a^n$.
3. $e^y = xy$.
4. $\log xy = x^2 + y^2$.
5. $x^y \cdot y^x = 1$.
6. $x^y + y^x = 1$.

59. Euler's Theorem.

If $u = Ax^\alpha y^\beta + Bx^{\alpha'} y^{\beta'} + \ldots = \Sigma Ax^\alpha y^\beta$, *say, where*

$$\alpha + \beta = \alpha' + \beta' = \ldots = n,$$

to show that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

By partial differentiation we obtain

$$\frac{\partial u}{\partial x} = \Sigma A\alpha x^{\alpha-1} y^\beta,$$

$$\frac{\partial u}{\partial y} = \Sigma A\beta x^\alpha y^{\beta-1},$$

then
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\Sigma A\alpha x^{\alpha}y^{\beta}+\Sigma A\beta x^{\alpha}y^{\beta}$$
$$=\Sigma A(\alpha+\beta)x^{\alpha}y^{\beta}$$
$$=n\Sigma Ax^{\alpha}y^{\beta}=nu.$$

It is clear that this theorem can be extended to the case of three or of any number of independent variables, and that if, for example,

$$u=Ax^{\alpha}y^{\beta}z^{\gamma}+Bx^{\alpha'}y^{\beta'}z^{\gamma'}+\ldots,$$

where $\alpha+\beta+\gamma=\alpha'+\beta'+\gamma'=\ldots=n,$

then will
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

The functions thus described are called *homogeneous functions of the* n^{th} *degree*, and the above result is known as Euler's Theorem on homogeneous functions.

EXAMPLES.

Verify Euler's theorem for the expressions:

$$(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^n+y^n),\qquad \frac{1}{x^2+xy+y^2},\qquad x^n\sin\frac{y}{x}.$$

EXAMPLES.

Find $\frac{dy}{dx}$ in the following cases:

1. $y=\frac{2+x^2}{1+x}$.
2. $y=\sqrt[n]{a+x}$.
3. $y=\sqrt[n]{a^2+x^2}$.
4. $y=\sqrt{\frac{1-x}{1+x}}$.
5. $y=\frac{1-x^2}{\sqrt{1+x^2}}$.
6. $y=\frac{x\sqrt{x^2-4a^2}}{\sqrt{x^2-a^2}}$.
7. $y=\sqrt{\frac{1-x}{1+x+x^2}}$.
8. $y=\log\frac{x^2+x+1}{x^2-x+1}$.
9. $y=\tan^{-1}(\log x)$.
10. $y=\sin x^{\circ}$.
11. $y=\sin(e^x)\log x$.
12. $y=\tan^{-1}(e^x)\log\cot x$.
13. $y=\log\cosh x$.
14. $y=\text{vers}^{-1}\log(\cot x)$.

15. $y = \cot^{-1}(\operatorname{cosec} x)$.

16. $y = \sin^{-1} \dfrac{1}{\sqrt{1+x^2}}$.

17. $y = \tan^{-1} \dfrac{1}{\sqrt{x^2-1}}$.

18. $y = (\sin^{-1} x)^m (\cos^{-1} x)^n$.

19. $y = \sin(e^x \log x) . \sqrt{1-(\log x)^2}$.

20. $y = \left(\dfrac{x}{n}\right)^{nx}\left(1 + \log \dfrac{x}{n}\right)$.

21. $y = b \tan^{-1}\left(\dfrac{x}{a} \tan^{-1} \dfrac{x}{a}\right)$.

22. $y = \dfrac{x \cos^{-1} x}{\sqrt{1-x^2}}$.

23. $y = \cos\left(a \sin^{-1} \dfrac{1}{x}\right)$.

24. $y = \sin^{-1} \dfrac{a + b\cos x}{b + a \cos x}$.

25. $y = e^{\tan^{-1} x} \log(\sec^2 x^3)$.

26. $y = e^{ax} \cos(b \tan^{-1} x)$.

27. $y = \tan^{-1}(a^{cx} . x^2)$.

28. $y = \sec(\log_a \sqrt{a^2 + x^2})$.

29. $y = \tan^{-1} x + \frac{1}{2} \log \dfrac{1+x}{1-x}$.

30. $y = \log(\log x)$.

31. $y = \log^n (x)$, where $\log^n$ means log log log...(repeated n times).

32. $y = \dfrac{1}{\sqrt{b^2 - a^2}} \log \dfrac{\sqrt{b+a} + \sqrt{b-a} \tan \dfrac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \tan \dfrac{x}{2}}$.

33. $y = \sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$.

34. $y = 10^{10^x}$.

35. $y = e^{e^x}$.

36. $y = e^{x^x}$.

37. $y = x^{e^x}$.

38. $y = x^{x^x}$.

39. $y = x^a + x^x$.

40. $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$.

41. $y = \tan^{-1}(a^{cx} x^{\sin x}) \dfrac{\sqrt{x}}{1+x^{\frac{3}{2}}}$.

42. $y = \sin^{-1}(e^{\tan^{-1} x})$.

43. $y = \sqrt{\left(1 + \cos \dfrac{m}{x}\right)\left(1 - \sin \dfrac{m}{x}\right)}$.

44. $y = \tan^{-1}\sqrt{\sqrt{x} + \cos^{-1} x}$.

45. $y = \left(\dfrac{1+\sqrt{x}}{1+2\sqrt{x}}\right)^{\sin e^{x^2}}$.

46. $y = (\cos x)^{\cot^2 x}$.

47. $y = (\cot^{-1} x)^{\frac{1}{x}}$.

48. $y = \left(1 + \dfrac{1}{x}\right)^x + x^{1+\frac{1}{x}}$.

49. $y = b \tan^{-1}\left(\dfrac{x}{a} + \tan^{-1} \dfrac{y}{x}\right)$.

50. $\tan y = e^{\cos^2 x}\sin x$. **51.** $ax^2+2hxy+by^2=1$.

52. $e^y = \dfrac{(a+bx^n)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(bx^n)^{\frac{1}{2}}}$. **53.** $(\cos x)^y=(\sin y)^x$.

54. $x=e^{\tan^{-1}\frac{y-x}{x^2}}$. **55.** $x=y\log xy$. **56.** $y=x^y$. **57.** $y=x^{y^x}$.

58. $y=x\log\dfrac{y}{a+bx}$. **59.** $ax^2+2hxy+by^2+2gx+2fy+c=0$.

60. $x^m y^n=(x+y)^{m+n}$. **61.** $y=e^{\tan^{-1}y}\log\sec^2 x^3$.

62. If $y=\dfrac{1}{2}\dfrac{a^2-b^2}{a^2+b^2}x\left\{\dfrac{p\sqrt[p]{x}}{p+1}+\dfrac{q\sqrt[q]{x}}{q+1}\right\}$, shew that when

$$x=\left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}} \text{ then will } \frac{dy}{dx}=\left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}.$$

63. Differentiate $\log_{10}x$ with regard to x^2.

64. Differentiate $(x^2+ax+a^2)^n\log\cot\dfrac{x}{2}$ with regard to

$\tan^{-1}(a\cos bx)$.

65. Differentiate $x^{\sin^{-1}x}$ with regard to $\sin^{-1}x$.

66. Differentiate $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with regard to $\tan^{-1}x$.

67. Differentiate $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ with regard to $\sqrt{1-x^4}$.

68. Differentiate $\sec^{-1}\dfrac{1}{2x^2-1}$ with regard to $\sqrt{1-x^2}$.

69. Differentiate $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ with regard to $\sec^{-1}\dfrac{1}{2x^2-1}$.

70. Differentiate $\tan^{-1}\dfrac{2x}{1-x^2}$ with regard to $\sin^{-1}\dfrac{2x}{1+x^2}$.

71. Differentiate $x^n\log\tan^{-1}x$ with regard to $\dfrac{\sin\sqrt{x}}{x^{\frac{1}{2}}}$.

72. If $y=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ prove $x\dfrac{dy}{dx}=\dfrac{y^2}{1-y\log x}$.

73. If $y=\dfrac{x}{1+\dfrac{x}{1+\dfrac{x}{1+\ldots\text{to }\infty}}}$, prove $\dfrac{dy}{dx}=\dfrac{1}{1+\dfrac{2x}{1+\dfrac{x}{1+\dfrac{x}{1+\dfrac{x}{1+\ldots}}}}}$.

74. If $y=x+\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x+\ldots\text{to }\infty}}}$, prove $\dfrac{dy}{dx}=\dfrac{1}{2-\dfrac{x}{x+\dfrac{1}{x+\dfrac{1}{x+\ldots}}}}$.

75. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\text{etc. to }\infty}}}}$,

$$y_1=\cos x/(2y-1).$$

76. If $S_n=$ the sum of a G.P. to n terms of which r is the common ratio, prove that

$$(r-1)\frac{dS_n}{dr}=(n-1)S_n-nS_{n-1}.$$

77. If $\dfrac{P}{Q}=a+\dfrac{1}{a_1+\dfrac{1}{a_2+\dfrac{1}{a_3+\ldots+\dfrac{1}{x}}}}$, prove $\dfrac{d}{dx}\left(\dfrac{P}{Q}\right)=\pm\dfrac{1}{Q^2}$.

78. Given $C=1+r\cos\theta+\dfrac{r^2\cos 2\theta}{2!}+\dfrac{r^3\cos 3\theta}{3!}+\ldots$

and $$S=r\sin\theta+\frac{r^2\sin 2\theta}{2!}+\frac{r^3\sin 3\theta}{3!}+\ldots$$

shew that $$C\frac{dC}{dr}+S\frac{dS}{dr}=(C^2+S^2)\cos\theta;$$

$$C\frac{dS}{dr}-S\frac{dC}{dr}=(C^2+S^2)\sin\theta.$$

79. If $y=\sec 4x$, prove that

$$\frac{dy}{dt}=\frac{16t(1-t^4)}{(1-6t^2+t^4)^2},\text{ where } t=\tan x.$$

80. If $y=e^{-x^z}\sec^{-1}(x\sqrt{z})$ and $z^4+x^2z=x^5$, find $\dfrac{dy}{dx}$ in terms of x and z.

81. Prove that if x be less than unity

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots\text{ ad inf.}=\frac{1}{1-x}.$$

82. Prove that if x be less than unity

$$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\dots \text{ ad inf.}=\frac{1+2x}{1+x+x^2}.$$

83. Given Euler's Theorem that

$$Lt_{n=\infty} \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\dots\cos\frac{x}{2^n}=\frac{\sin x}{x},$$

prove $\quad \frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\frac{1}{2^3}\tan\frac{x}{2^3}+\dots \text{ ad inf.}=\frac{1}{x}-\cot x,$

and $\quad \frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{2^4}\sec^2\frac{x}{2^2}+\frac{1}{2^6}\sec^2\frac{x}{2^3}+\dots \text{ ad inf.}=\operatorname{cosec}^2 x-\frac{1}{x^2}.$

84. Differentiate logarithmically the expressions for $\sin\theta$ and $\cos\theta$ in factors, and deduce the sums to infinity of the following series

(*a*) $\quad \dfrac{1}{\theta^2-\pi^2}+\dfrac{1}{\theta^2-2^2\pi^2}+\dfrac{1}{\theta^2-3^2\pi^2}+\dfrac{1}{\theta^2-4^2\pi^2}+\dots\dots$

(*b*) $\quad \dfrac{1}{1^2+x^2}+\dfrac{1}{2^2+x^2}+\dfrac{1}{3^2+x^2}+\dfrac{1}{4^2+x^2}+\dots\dots$

(*c*) $\quad \dfrac{1}{1^2+x^2}+\dfrac{1}{3^2+x^2}+\dfrac{1}{5^2+x^2}+\dfrac{1}{7^2+x^2}+\dots\dots$

(*d*) $\quad 1+\dfrac{2}{1+1^2}+\dfrac{2}{1+2^2}+\dfrac{2}{1+3^2}+\dots\dots$

85. Sum to infinity the series

$$\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1+x^{\frac{1}{2}}}+\frac{1}{4}\cdot\frac{1}{1+x^{\frac{1}{4}}}+\frac{1}{8}\cdot\frac{1}{1+x^{\frac{1}{8}}}+\dots\dots$$

86. If H_n represent the sum of the homogeneous products of n dimensions of x, y, z, prove

(*a*) $\quad x\dfrac{\partial H_n}{\partial x}+y\dfrac{\partial H_n}{\partial y}+z\dfrac{\partial H_n}{\partial z}=nH_n;$

(*b*) $\quad \dfrac{\partial H_n}{\partial x}+\dfrac{\partial H_n}{\partial y}+\dfrac{\partial H_n}{\partial z}=(n+2)H_{n-1}.$

CHAPTER V.

SUCCESSIVE DIFFERENTIATION.

60. WHEN y is a given function of x, and $\frac{dy}{dx}$ has been found, we may proceed to differentiate a second time obtaining $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. This expression is called the *second differential coefficient* of y with respect to x. We may then differentiate again and obtain the *third* differential coefficient and so on.

The expression $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is abbreviated into $\left(\frac{d}{dx}\right)^2 y$ or $\frac{d^2y}{dx^2}$; $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is written $\frac{d^3y}{dx^3}$; and so on.

Thus the several differential coefficients of y are written

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\dots \frac{d^ny}{dx^n}\dots.$$

They are often further abbreviated into

$$y_1,\ y_2,\ y_3,\dots y_n\dots.$$

Ex. 1. Thus if $y=x^n$, we have

$$y_1=nx^{n-1},$$
$$y_2=n(n-1)x^{n-2},$$
$$y_3=n(n-1)(n-2)x^{n-3},$$

and generally $\quad y_r=n(n-1)\dots(n-r+1)x^{n-}$

$$y_{n+1}=y_{n+2}=y_{n+3}=\dots=0.$$

Ex. 2. If
$$\begin{aligned} y &= \tan x, \\ y_1 &= \sec^2 x = 1 + y^2, \\ y_2 &= 2yy_1 = 2(y + y^3), \\ y_3 &= 2(1+3y^2)\,y_1 = 2(1+4y^2+3y^4), \\ y_4 &= 2(8y+12y^3)\,y_1 = 8(2y+5y^3+3y^5), \\ &\text{\&c.} \end{aligned}$$

Ex. 3. If
$$y = (\sin^{-1} x)^2,$$
$$y_1 = 2(\sin^{-1} x)/\sqrt{1-x^2},$$
$\therefore$ squaring, $$(1-x^2)\,y_1^2 = 4y.$$
Hence differentiating, $(1-x^2)\,2y_1y_2 - 2xy_1^2 = 4y_1$,
and dividing by $2y_1$, $\quad (1-x^2)\,y_2 - xy_1 = 2.$

61. Standard results and processes.

The n^{th} differential coefficient of some functions are easy to find.

Ex. 1. If $y = e^{ax}$ we have $y_1 = ae^{ax}$, $y_2 = a^2e^{ax}$, ...
$$y_n = a^n e^{ax}.$$

Cor. i. If $a = 1$,
$$y = e^x,\ y_1 = e^x,\ y_2 = e^x,\ \ldots\ y_n = e^x.$$

Cor. ii.
$$\begin{aligned} y &= a^x = e^{x\log_e a}; \\ y_1 &= (\log_e a)\,e^{x\log_e a} = (\log_e a)\,a^x; \\ y_2 &= (\log_e a)^2\,e^{x\log_e a} = (\log_e a)^2\,a^x; \\ \text{etc.} &= \text{etc.} \\ y_n &= (\log_e a)^n\,e^{x\log_e a} = (\log_e a)^n\,a^x. \end{aligned}$$

Ex. 2. If $y = \log_e(x+a)$;
$$y_1 = \frac{1}{x+a};\quad y_2 = -\frac{1}{(x+a)^2};\quad y_3 = \frac{(-1)(-2)}{(x+a)^3};\ .$$
$$y_n = \frac{(-1)(-2)(-3)\ldots(-n+1)}{(x+a)^n}$$
$$= \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}.$$

Cor. If $\quad y = \dfrac{1}{x+a},\quad y_n = \dfrac{(-1)^n\,n!}{(x+a)^{n+1}}.$

Ex. 3. If $y=\sin(ax+b)$;

$$y_1=a\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right);$$

$$y_2=a^2\sin\left(ax+b+\frac{2\pi}{2}\right);$$

$$y_3=a^3\sin\left(ax+b+\frac{3\pi}{2}\right);$$

$$\dots\dots\dots\dots\dots\dots$$

$$y_n=a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

Similarly, if $y=\cos(ax+b)$,

$$y_n=a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

Cor. If $a=1$ and $b=0$;

then, when $y=\sin x$, $y_n=\sin\left(x+\frac{n\pi}{2}\right)$;

and, when $y=\cos x$, $y_n=\cos\left(x+\frac{n\pi}{2}\right)$.

Ex. 4. If $y=e^{ax}\sin(bx+c)$;

$$y_1=ae^{ax}\sin(bx+c)+be^{ax}\cos(bx+c).$$

Let $a=r\cos\phi$ and $b=r\sin\phi$,

so that a^2+b^2 and $\tan\phi=\frac{b}{a}$;

and therefore $y_1=re^{ax}\sin(bx+c+\phi)$.

Thus the operation of differentiating this expression is equivalent to multiplying by r and adding ϕ to the angle.

Thus $y_2=r^2e^{ax}\sin(bx+c+2\phi)$,

and generally $y_n=r^ne^{ax}\sin(bx+c+n\phi)$.

Similarly, if $y=e^{ax}\cos(bx+c)$,

$$y_n=r^ne^{ax}\cos(bx+c+n\phi).$$

These results are often wanted and the student should be able to obtain them immediately.

Ex. 5. Find the n^{th} differential coefficient of $\sin^3 x$.

We have $y=\sin^3 x=\frac{1}{4}(3\sin x-\sin 3x)$.

Hence $$y_n=\frac{1}{4}\left\{3\sin\left(x+\frac{n\pi}{2}\right)-3^n\sin\left(3x+\frac{n\pi}{2}\right)\right\}.$$

Ex. 6. If $y=\sin^2 x\cos^3 x$, find y_n.

Here
$$y=\frac{1}{4}\sin^2 2x\cos x=\frac{1}{8}(1-\cos 4x)\cos x$$
$$=\frac{1}{16}(2\cos x-\cos 3x-\cos 5x),$$

and
$$y_n=\frac{1}{16}\left\{2\cos\left(x+\frac{n\pi}{2}\right)-3^n\cos\left(3x+\frac{n\pi}{2}\right)-5^n\cos\left(5x+\frac{n\pi}{2}\right)\right\}.$$

EXAMPLES.

Find y_n in the following cases:

1. $\dfrac{1}{ax+b}$.
2. $\dfrac{1}{a-x}$.
3. $\dfrac{1}{a-bx}$.
4. $\dfrac{x}{a+bx}$.
5. $\dfrac{ax+b}{cx+d}$.
6. $\dfrac{x^2}{x-a}$.
7. $\dfrac{1}{(x+a)^4}$.
8. $\sqrt{x+a}$.
9. $(x+a)^{-\frac{3}{5}}$.
10. $\log(ax+b)^p$.
11. $y=\sin x\sin 2x$.
12. $y=e^x\sin x\sin 2x$.
13. $y=e^x\sin^2 x$.
14. $y=e^{ax}\cos^2 bx$.
15. $y=\sin x\sin 2x\sin 3x$.
16. $y=e^{3x}\sin^2 x\cos^3 x$.
17. $y=\sin^2 x\sin 2x$.
18. $y=e^x\sin^2 x\sin 2x$.

62. Use of Partial Fractions.

Fractional expressions whose numerators and denominators are both rational algebraic expressions are differentiated n times by first putting them into partial fractions.

Ex. 1.
$$y=\frac{x^2}{(x-a)(x-b)(x-c)}=\frac{a^2}{(a-b)(a-c)}\frac{1}{x-a}$$
$$+\frac{b^2}{(b-c)(b-a)}\frac{1}{x-b}+\frac{c^2}{(c-a)(c-b)}\frac{1}{x-c}$$

(see note on partial fractions Art. 66);

therefore
$$y_n=\frac{a^2}{(a-b)(a-c)}\frac{(-1)^n n!}{(x-a)^{n+1}}+\frac{b^2}{(b-c)(b-a)}\frac{(-1)^n n!}{(x-b)^{n+1}}$$
$$+\frac{c^2}{(c-a)(c-b)}\frac{(-1)^n n!}{(x-c)^{n+1}}.$$

Ex. 2. $$y=\frac{x^2}{(x-1)^2(x+2)}.$$

To put this into Partial Fractions let $x=1+z$;

then $$y=\frac{1}{z^2}\cdot\frac{1+2z+z^2}{3+z}$$

$$=\frac{1}{z^2}\left(\frac{1}{3}+\frac{5z}{9}+\frac{4}{9}\frac{z^2}{3+z}\right) \text{ by division}$$

$$=\frac{1}{3z^2}+\frac{5}{9z}+\frac{4}{9}\frac{1}{3+z}$$

$$=\frac{1}{3(x-1)^2}+\frac{5}{9(x-1)}+\frac{4}{9(x+2)},$$

whence $$y_n=\frac{(n+1)!\,(-1)^n}{3(x-1)^{n+2}}+\frac{5n!\,(-1)^n}{9(x-1)^{n+1}}+\frac{4n!\,(-1)^n}{9(x+2)^{n+1}}.$$

63. Application of Demoivre's Theorem.

When quadratic factors which are not resolvable into real linear factors occur in the denominator, it is often convenient to make use of Demoivre's Theorem as in the following example.

Let $$y=\frac{1}{x^2+a^2}=\frac{1}{(x+\iota a)(x-\iota a)}$$

$$=\frac{1}{2\iota a}\left\{\frac{1}{x-\iota a}-\frac{1}{x+\iota a}\right\}.$$

Then $$y_n=\frac{1}{2\iota a}(-1)^n n!\left\{\frac{1}{(x-\iota a)^{n+1}}-\frac{1}{(x+\iota a)^{n+1}}\right\}.$$

Let $$x=r\cos\theta \text{ and } a=r\sin\theta,$$

whence $$r^2=x^2+a^2 \text{ and } \tan\theta=\frac{a}{x}.$$

Hence $$y_n=\frac{(-1)^n n!}{2\iota a r^{n+1}}\{(\cos\theta-\iota\sin\theta)^{-n-1}-(\cos\theta+\iota\sin\theta)^{-n-1}\}$$

$$=\frac{(-1)^n n!}{2\iota a r^{n+1}}.2\iota\sin(n+1)\theta$$

$$=\frac{(-1)^n n!}{a^{n+2}}\sin(n+1)\theta\sin^{n+1}\theta,$$

where $$\theta=\tan^{-1}\frac{a}{x}.$$

COR. 1. Similarly if $y=\dfrac{1}{(x+b)^2+a^2}$,

$$y_n=\frac{(-1)^n n!}{a^{n+2}}\sin(n+1)\theta\sin^{n+1}\theta,$$

where $$\theta=\tan^{-1}\frac{a}{b+x}.$$

COR. 2. If $$y=\tan^{-1}\frac{x}{a},\quad y_1=\frac{a}{x^2+a^2},$$

and $$y_n=\frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta,$$

where $$\tan\theta=\frac{a}{x}=\cot y.$$

EXAMPLES.

Find the n^{th} differential coefficients of y with respect to x in the following cases:

1. $y=\dfrac{1}{4x^2-1}$.
2. $y=\dfrac{1}{4x^2+1}$.
3. $y=\frac{1}{2}\log\dfrac{x+a}{x-a}$.
4. $y=\dfrac{1}{x^4-a^4}$.
5. $y=\dfrac{1}{(x^2-a^2)(x^2-b^2)}$.
6. $y=\dfrac{1}{(x^2+a^2)(x^2+b^2)}$.
7. $y=\tan^{-1}\dfrac{2x}{1-x^2}$.
8. $y=\dfrac{1}{x^2+x+1}$.
9. $y=\dfrac{x}{x^4+x^2+1}$.
10. $y=\dfrac{x}{x^4+x^3+2x^2+x+1}$.

64. Leibnitz's Theorem.

[Lemma. If ${}_nC_r$ denote the number of combinations of n things r at a time then will

$${}_nC_r+{}_nC_{r+1}={}_{n+1}C_{r+1}.$$

For $$\frac{|n}{|r\,|n-r}+\frac{|n}{|r+1\,|n-r-1}=\frac{|n}{|r\,|n-r-1}\left\{\frac{1}{n-r}+\frac{1}{r+1}\right\}$$

$$=\frac{|n+1}{|r+1\,|n-r}={}_{n+1}C_{r+1}.]$$

Let $y = uv$, and let suffixes denote differentiations with regard to x. Then

$$y_1 = u_1 v + u v_1,$$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2, \text{ by differentiation.}$$

Assume generally that

$$y_n = u_n v + {}_nC_1 u_{n-1} v_1 + {}_nC_2 u_{n-2} v_2 + \dots$$
$$+ {}_nC_r u_{n-r} v_r + {}_nC_{r+1} u_{n-r-1} v_{r+1} + \dots + u v_n \dots (\alpha).$$

Therefore differentiating

$$y_{n+1} = u_{n+1} v + u_n v_1 \begin{Bmatrix} {}_nC_1 \\ +1 \end{Bmatrix} + u_{n-1} v_2 \begin{Bmatrix} {}_nC_2 \\ + {}_nC_1 \end{Bmatrix} + \dots$$
$$+ u_{n-r} v_{r+1} \begin{Bmatrix} {}_nC_{r+1} \\ + {}_nC_r \end{Bmatrix} + \dots + u v_{n+1}$$
$$= u_{n+1} v + {}_{n+1}C_1 u_n v_1 + {}_{n+1}C_2 u_{n-1} v_2 + {}_{n+1}C_3 u_{n-2} v_3 + \dots$$
$$+ {}_{n+1}C_{r+1} u_{n-r} v_{r+1} + \dots + u v_{n+1}, \text{ by the Lemma;}$$

therefore if the law (α) hold for n differentiations it holds for $n+1$.

But it was proved to hold for two differentiations, and therefore it holds for three; therefore for four; and so on; and therefore it is generally true, *i.e.*,

$$(uv)_n = u_n v + {}_nC_1 u_{n-1} v_1 + {}_nC_2 u_{n-2} v_2 + \dots$$
$$+ {}_nC_r u_{n-r} v_r + \dots + u v_n.$$

65. Applications.

Ex. 1. $y = x^3 \sin ax$.

Here we take $\sin ax$ as u and x^3 as v.

Now $v_1 = 3x^2$, $v_2 = 3 \cdot 2x$, $v_3 = 3 \cdot 2$, and v_4 &c. are all zero.

Also
$$u_n = a^n \sin\left(ax + \frac{n\pi}{2}\right), \text{ etc.}$$

Hence by Leibnitz's Theorem we have

$$y_n = x^3 a^n \sin\left(ax + \frac{n\pi}{2}\right) + n 3x^2 a^{n-1} \sin\left(ax + \frac{n-1}{2}\pi\right)$$

$$+\frac{n(n-1)}{2!}3.2xa^{n-2}\sin\left(ax+\frac{n-2}{2}\pi\right)$$
$$+\frac{n(n-1)(n-2)}{3!}3.2.1a^{n-3}\sin\left(ax+\frac{n-3}{2}\pi\right)$$

The student will note that if one of the factors be a power of x it will be advisable to take that factor as v.

Ex. 2. Let $y=x^4.e^{ax}$; find y_5.

Here $$v=x^4, \quad u=e^{ax},$$
so that $v_1=4x^3$, $v_2=12x^2$, $v_3=24x$, $v_4=24$, and v_5 etc. all vanish.

Also $$u_n=a^ne^{ax}, \text{ etc.}$$
whence
$$y_5=a^5e^{ax}x^4+5a^4e^{ax}.4x^3+10.a^3e^{ax}.12x^2+10a^2e^{ax}.24x+5ae^{ax}.24$$
$$=ae^{ax}\{a^4x^4+20a^3x^3+120a^2x^2+240ax+120\}.$$

Ex. 3. Differentiate n times the equation
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

$$\frac{d^n}{dx^n}(x^2y_2)=x^2y_{n+2}+n.2x.y_{n+1}+\frac{n(n-1)}{2!}2y_n,$$
$$\frac{d^n}{dx^n}(xy_1)= \qquad xy_{n+1}+ \qquad ny_n,$$
$$\frac{d^ny}{dx^n}= \qquad y_n;$$

therefore by addition
$$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0,$$
or $$x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^ny}{dx^n}=0.$$

Ex. 4. Even when the general value of y_n cannot be obtained we may sometimes find its value for $x=0$ as follows.

Suppose $$y=[\log(x+\sqrt{1+x^2})]^2,$$
then $$y_1=2\log(x+\sqrt{1+x^2})/\sqrt{1+x^2} \quad \ldots\ldots\ldots\ldots (1),$$
and $$(1+x^2)y_1^2=4y,$$
whence differentiating and dividing by $2y_1$,
$$(1+x^2)y_2+xy_1=2 \quad \ldots\ldots\ldots\ldots (2).$$

Differentiating n times by Leibnitz's Theorem

$$(1+x^2)\,y_{n+2}+2nxy_{n+1}+n\,(n-1)\,y_n$$
$$+xy_{n+1}+ny_n=0$$

or
$$(1+x^2)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0.$$

Putting $x=0$ we have

$$(y_{n+2})_0=-n^2\,(y_n)_0 \qquad \ldots\ldots(3),$$

indicating by suffix zero the value attained upon the vanishing of x.

Now, when $x=0$ we have from the value of y and equations (1) and (2)

$$(y)_0=0,\ (y_1)_0=0,\ (y_2)_0=2.$$

Hence equation (3) gives

$$(y_3)_0=(y_5)_0=(y_7)_0 \ \ldots\ldots =(y_{2k+1})_0=0$$

and
$$(y_4)_0=-2^2\,.\,2,$$
$$(y_6)_0=4^2\,.\,2^2\,.\,2,$$
$$(y_8)_0=-6^2\,.\,4^2\,.\,2^2\,.\,2,$$
etc.,
$$(y_{2k})_0=(-1)^{k-1}2\,.\,2^2\,.\,4^2\,.\,6^2\ldots\ldots(2k-2)^2$$
$$=(-1)^{k-1}2^{2k-1}\{(k-1)!\}^2.$$

EXAMPLES.

Apply Leibnitz's Theorem to find y_n in the following cases:

1. $y=xe^x$.
2. $y=x^2e^{ax}$.
3. $y=x^2\log x$.
4. $y=x^2\sin x$.
5. $y=e^{ax}.\sin bx$.
6. $y=\dfrac{x^n}{1+x}$.
7. $y=x\tan^{-1}x$.
8. $y=x^2\tan^{-1}x$.

66. Note on Partial Fractions.

Since a number of examples on successive differentiation and on integration depend on the ability of the student to put certain fractional forms into partial fractions, we give the methods to be pursued in a short note.

Let $\dfrac{f(x)}{\phi(x)}$ be the fraction which is to be resolved into its partial fractions.

1. If $f(x)$ be not already of lower degree than the denominator, *we can divide out until the numerator of the remaining fraction is of lower degree: e.g.*

$$\frac{x^2}{(x-1)(x-2)} = 1 + \frac{3x-2}{(x-1)(x-2)}.$$

Hence we shall consider only the case in which $f(x)$ is of lower degree than $\phi(x)$.

2. If $\phi(x)$ contain a single factor $(x-a)$, not repeated, we proceed thus: suppose

$$\phi(x) = (x-a)\,\psi(x),$$

and let
$$\frac{f(x)}{(x-a)\,\psi(x)} = \frac{A}{x-a} + \frac{\chi(x)}{\psi(x)},$$

A being independent of x.

Hence
$$\frac{f(x)}{\psi(x)} \equiv A + (x-a)\frac{\chi(x)}{\psi(x)}.$$

This is an identity and therefore true for all values of the variable x; put $x=a$. Then, since $\psi(x)$ does not vanish when $x=a$ (for by hypothesis $\psi(x)$ does not contain $x-a$ as a factor), we have

$$A = \frac{f(a)}{\psi(a)}.$$

Hence the rule to find A is, "Put $x=a$ in every portion of the fraction except in the factor $x-a$ itself."

Ex. (i)
$$\frac{x-c}{(x-a)(x-b)} = \frac{a-c}{a-b}\cdot\frac{1}{x-a} + \frac{b-c}{b-a}\cdot\frac{1}{x-b}.$$

Ex. (ii)
$$\frac{x^2+px+q}{(x-a)(x-b)(x-c)} = \frac{a^2+pa+q}{(a-b)(a-c)}\,\frac{1}{x-a}$$
$$+ \frac{b^2+pb+q}{(b-c)(b-a)}\,\frac{1}{x-b} + \frac{c^2+pc+q}{(c-a)(c-b)}\,\frac{1}{x-c}.$$

Ex. (iii)
$$\frac{\quad}{(x-1)(x-2)(x-3)} = \frac{\quad}{2(x-1)} - \frac{2}{x-2} + \frac{3}{2(x-3)}$$

Ex. (iv)
$$\frac{\quad}{(x-a)(x-b)}.$$

Here the numerator not being of *lower degree than the denominator*, we divide the numerator by the denominator. The result will then be expressible in the form $1 + \frac{A}{x-a} + \frac{B}{x-b}$, where A and B are found as before and are respectively $\frac{a^2}{a-b}$ and $\frac{b^2}{b-a}$.

3. Suppose the factor $(x-a)$ in the denominator to be repeated r times so that

$$\phi(x)=(x-a)^r\psi(x).$$

Put $$x-a=y.$$

Then $$\frac{f(x)}{\phi(x)}=\frac{f(a+y)}{y^r\psi(a+y)},$$

or expanding each function by any means in ascending powers of y,

$$=\frac{A_0+A_1y+A_2y^2+\dots}{y^r(B_0+B_1y+B_2y^2+\dots)}.$$

Divide out thus:—

$$B_0+B_1y+\dots\,)\,A_0+A_1y+\dots\,(\,C_0+C_1y+C_2y^2+\dots,$$
$$\text{etc.,}$$

and let the division be continued until y^r is a factor of the remainder.

Let the remainder be $y^r\chi(y)$.

Hence the fraction $$=\frac{C_0}{y^r}+\frac{C_1}{y^{r-1}}+\frac{C_2}{y^{r-2}}+\dots+\frac{C_{r-1}}{y}+\frac{\chi(y)}{\psi(a+y)}$$

$$=\frac{C_0}{(x-a)^r}+\frac{C_1}{(x-a)^{r-1}}+\frac{C_2}{(x-a)^{r-2}}+\dots$$
$$+\frac{C_{r-1}}{x-a}+\frac{\chi(x-a)}{\psi(x)}.$$

Hence the partial fractions corresponding to the factor $(x-a)^r$ are determined by a long division sum.

Ex. Take $$\frac{\quad}{(x-1)^3(x+1)}.$$

Put $$x-1=y.$$

Hence the fraction $$=\frac{(1+y)^2}{y^3(2+y)}.$$

$$
\begin{array}{r|l}
2+y & 1+2y+y^2\ \left(\tfrac12+\tfrac34y+\tfrac18y^2-\tfrac18\,\dfrac{y^3}{2+y}\right. \\
 & \underline{1+\tfrac12y} \\
 & \quad \tfrac32y+y^2 \\
 & \quad \underline{\tfrac32y+\tfrac34y^2} \\
 & \qquad \tfrac14y^2 \\
 & \qquad \underline{\tfrac14y^2+\tfrac18y^3} \\
 & \qquad\quad -\tfrac18y^3
\end{array}
$$

Therefore the fraction

$$=\frac{1}{2y^3}+\frac{3}{4y^2}+\frac{1}{8y}-\frac{1}{8(2+y)}$$

$$=\frac{1}{2(x-1)^3}+\frac{3}{4(x-1)^2}+\frac{1}{8(x-1)}-\frac{1}{8(x+1)}.$$

4. If a factor, such as x^2+ax+b, which is not resolvable into real linear factors occur in the denominator, the form of the corresponding partial fraction is $\dfrac{Ax+B}{x^2+ax+b}$. For instance, if the expression be

$$\frac{}{(x-a)(x-b)^2(x^2+a^2)(x^2+b^2)^2}$$

the proper assumption for the form in partial fractions would be

$$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{(x-b)^2}+\frac{Dx+E}{x^2+a^2}+\frac{Fx+G}{x^2+b^2}+\frac{Hx+K}{(x^2+b^2)^2},$$

where A, B, and C can be found according to the preceding methods, and on reduction to a common denominator we can, by equating coefficients of like powers in the two numerators, find the remaining letters D, E, F, G, H, K. Variations upon these methods will suggest themselves to the student.

EXAMPLES.

1. Given $y=\sin x^2$, find y_2, y_3, y_4.

2. Given $y=x\sin x$, find y_2, y_3, y_4.

3. Given $y=e^x\sin x$, find y_2, ... y_6.

4. Given $y=x^3e^{ax}$, find y_3 and y_n.

5. If $y=Ae^{mx}+Be^{-mx}$, prove $y_2=m^2y$.

6. If $y=A\sin mx+B\cos mx$, prove $y_2=-m^2y$.

7. If $y=a\sin\log x$, prove $x^2y_2+xy_1+y=0$.

8. If $y=\log\left(\dfrac{x}{a+bx}\right)^x$, prove $x^3y_2=(y-xy_1)^2$.

9. If $y=A(x+\sqrt{x^2-1})^n+B(x-\sqrt{x^2-1})^n$,

prove $$(x^2-1)y_2+xy_1-n^2y=0.$$

10. If $$y=\frac{(x-a)(x-b)}{(x-c)(x-d)},\text{ find } y_n.$$

11. If $$y=\frac{1}{(x-1)^3(x-2)},\text{ find } y_n.$$

12. If $$y=x^n\log x,\text{ find } y_2,\ y_3,\ y_n,\ y_{n+1}.$$

13. If $$x=\cosh\left(\frac{1}{m}\log y\right)$$

prove $$(x^2-1)y_2+xy_1-m^2y=0,$$

and $$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

14. Find y_n if $$y=\frac{1}{x^3+1}.$$

15. Find y_n if $$y=\frac{1}{(x+1)(x^2+1)}.$$

16. Find y_n if $$y=\frac{x^2}{(x-1)^3(x+1)}.$$

17. Prove that if $y=\sin(m\sin^{-1}x)$,
$$(1-x^2)y_2=xy_1-m^2y,$$
and $$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n.$$
Hence shew that
$$Lt_{x=0}\frac{y_{n+2}}{y_n}=n^2-m^2.$$

18. If $y=e^{\tan^{-1}x}$, prove that
$$(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0.$$

19. If $y=e^{a\sin^{-1}x}$, prove that
$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0,$$
and $$Lt_{x=0}\frac{y_{n+2}}{y_n}=n^2+a^2.$$

20. If $$u=\sin nx+\cos nx,$$
$$u_r=n^r\{1+(-1)^r\sin 2nx\}^{\frac{1}{2}}.$$

21. If $$y=e^{ax}\{a^2x^2-2nax+n(n+1)\},$$
$$y_n=a^{n+2}x^2e^{ax}.$$

22. If $$x\cos\theta+y\sin\theta=a,$$
and $$x\sin\theta-y\cos\theta=b,$$
prove that $$\frac{d^px}{d\theta^p}\cdot\frac{d^qy}{d\theta^q}-\frac{d^qx}{d\theta^q}\cdot\frac{d^py}{d\theta^p}$$
is constant.

23. Prove that
$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\Big[P\sin\left(x+\frac{n\pi}{2}\right)+Q\cos\left(x+\right.$$

where
$$P=x^n-n(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4}-\ldots,$$
and $Q=nx^{n-1}-n(n-1)(n-2)x^{n-3}+\ldots.$

24. Prove

$$\frac{d^n}{dx^n}\left(\frac{\cos x}{x}\right)=\left[P\cos\left(x+\frac{n\pi}{2}\right)-Q\sin\left(x+\frac{n\pi}{2}\right)\right]/x^{n+1},$$

where P and Q have the same values as in **23**.

25. Prove that

$$\frac{d^n}{dx^n}\left(\frac{e^{ax}\sin bx}{x}\right)=e^{ax}\{P\sin(bx+n\phi)+Q\cos(bx+n\phi)\}/x^{n+1},$$

where

$$P=(rx)^n-n(rx)^{n-1}\cos\phi+n(n-1)(rx)^{n-2}\cos 2\phi-\dots,$$
$$Q=\qquad n(rx)^{n-1}\sin\phi-n(n-1)(rx)^{n-2}\sin 2\phi+\dots,$$

$r^2=a^2+b^2$, and $\tan\phi=b/a$.

26. Prove that

$$\frac{d^n}{dx^n}(x^n\sin x)=n!\,(P\sin x+Q\cos x),$$

where $$P=1-{}^nC_2\frac{x^2}{2!}+{}^nC_4\frac{x^4}{4!}-\dots$$

and $$Q={}^nC_1x-{}^nC_3\frac{x^3}{3!}+{}^nC_5\frac{x^5}{5!}-\dots.$$

27. Shew that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x^m}\right)$$
$$=\frac{(-1)^n n!}{(m-1)!\,x^{m+n}}\left[\frac{(m+n-1)!}{n!}\log x-\sum_{r=0}^{r=n-1}\left\{\frac{(m+r-1)!}{r!\,(n-r)}\right\}\right].$$

[I. C. S., 1892.]

CHAPTER VI.

EXPANSIONS.

67. THE student will have already met with several expansions of given explicit functions in ascending integral powers of the independent variable; for example, those tabulated on pages 10 and 11, which occur in ordinary Algebra and Trigonometry.

The principal methods of development in common use may be briefly classified as follows:

I. By purely Algebraical or Trigonometrical processes.

II. By Taylor's or Maclaurin's Theorems.

III. By the use of a differential equation.

IV. By Differentiation of a known series, or a converse process.

These methods we proceed to explain and exemplify.

68. METHOD I. **Algebraic and Trigonometrical Methods.**

Ex. 1. Find the first three terms of the expansion of $\log \sec x$ **in ascending powers of** x.

By Trigonometry

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Hence $\log \sec x = -\log \cos x = -\log(1-z)$,

where $z = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots;$

and expanding log $(1-z)$ by the logarithmic theorem we obtain

$$\log \sec x = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

$$= \left[\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right] + \frac{1}{2}\left[\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right]^2$$

$$+ \frac{1}{3}\left[\frac{x^2}{2!} \quad \ldots\right]^3 \ldots$$

$$= \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} -$$

$$+ \frac{x^4}{8} - \frac{x^6}{48} + \ldots$$

$$+ \frac{x^6}{24} - \ldots;$$

hence $\qquad \log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} \ldots$

Ex. 2. Expand $\cos^3 x$ in powers of x.

Since $\quad 4\cos^3 x = \cos 3x + 3\cos x$

$$= 1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \ldots + (-1)^n \frac{3^{2n} x^{2n}}{(2n)!} + \ldots$$

$$+ 3\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\right],$$

we obtain $\quad \cos^3 x = \frac{1}{4}\left\{(1+3) - (3^2+3)\frac{x^2}{2!} + (3^4+3)\frac{x^4}{4!} - \ldots\right.$

$$\left. + (-1)^n (3^{2n}+3)\frac{x^{2n}}{(2n)!} + \ldots\right\}.$$

Similarly $\quad \sin^3 x = \frac{1}{4}\left\{(3^3-3)\frac{x^3}{3!} - (3^5-3)\frac{x^5}{5!} + (3^7-3)\frac{x^7}{7!} - \ldots\right.$

$$\left. + (-1)^n \frac{3^{2n-1}-3}{(2n-1)!} x^{2n-1} - \ldots\right\}.$$

Ex. 3. Expand $\tan x$ in powers of x as far as the term involving x^5.

Since $\qquad \tan x = \dfrac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots}$

we may by actual division show that

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

Ex. 4. Expand $\frac{1}{2}\{\log(1+x)\}^2$ in powers of x.

Since
$$(1+x)^y \equiv e^{y\log(1+x)},$$
we have, by expanding each side of this identity,

$$1+yx+\frac{y(y-1)}{2!}x^2+\frac{y(y-1)(y-2)}{3!}x^3+\frac{y(y-1)(y-2)(y-3)}{4!}x^4+\dots$$
$$\equiv 1+y\log(1+x)+\frac{y^2}{2!}\{\log(1+x)\}^2+\dots$$

Hence, equating coefficients of y^2,

$$\frac{1}{2}\{\log(1+x)\}^2=\frac{x^2}{2!}-\frac{1+2}{3!}x^3+\frac{1.2+2.3+3.1}{4!}x^4-\text{etc.},$$

a series which may be written in the form

$$\frac{x^2}{2}-\left(1+\frac{1}{2}\right)\frac{x^3}{3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4}-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^5}{5}+\dots$$

EXAMPLES.

1. Prove $\qquad e^{x\sin x}=1+x^2+\frac{1}{3}x^4+\frac{1}{120}x^6\dots.$

2. Prove $\qquad \cosh^n x=1+\frac{nx^2}{2!}+n(3n-2)\frac{x^4}{4!}.$

3. Prove $\qquad \log\frac{\sin x}{x}=-\frac{x^2}{6}-\frac{x^4}{180}\dots.$

4. Prove $\qquad \log\frac{\sinh x}{x}=\frac{x^2}{6}-\frac{x^4}{180}\dots.$

5. Prove $\qquad \log x\cot x=-\frac{x^2}{3}-\frac{7}{90}x^4\dots.$

6. Prove $\qquad \log\frac{\tan^{-1}x}{x}=-\frac{x^2}{3}+\frac{13}{90}x^4-\frac{251}{5.7.9^2}x^6\dots.$

7. Prove
$$\log(1-x+x^2)=-x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{3}-\frac{x^7}{7}+\frac{x^8}{8}\dots.$$

8. Expand $\log(1+x^3e^x)$ as far as the term containing x^6.

9. Expand in powers of x,

(*a*) $\tan \quad q+px$. (*c*) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(*b*) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$. (*d*) $\cos^{-1}\dfrac{x-x^{-}}{x+x^{-}}$.

69. Method II. Taylor's and Maclaurin's Theorems.

It has been discovered that the Binomial, Exponential, and other well-known expansions are all particular cases of one general theorem, which has for its object the *expansion of $f(x+h)$ in ascending integral positive powers of h*, $f(x)$ being a function of x *of any form whatever*. It is found that such an expansion is not always possible, but the student is referred to a later chapter for a rigorous discussion of the limitations of the Theorem.

70. Taylor's Theorem.

The theorem referred to is that *under certain circumstances*

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\dots$$
$$+\frac{h^n}{n!}f^{(n)}(x)+\dots \text{ to infinity},$$

an expansion of $f(x+h)$ in powers of h.

This is known as Taylor's Theorem.

Assuming the possibility of expanding $f(x+h)$ in a *convergent* series of *positive integral* powers of h, let

$$f(x+h)=A_0+A_1h+A_2\frac{h^2}{2!}+A_3\frac{h^3}{3!}+\dots\dots(1),$$

where $A_0, A_1, A_2, \dots$ are functions of x alone which are to be determined.

Now $\dfrac{df(x+h)}{dh} = \dfrac{df(x+h)}{d(x+h)} \cdot \dfrac{d(x+h)}{dh} = f'(x+h)$,

for x and h are independent quantities and therefore x may be considered constant in differentiating with regard to h, so that $\dfrac{d(x+h)}{dh} = 1$.

Similarly

$$\frac{d^2 f(x+h)}{dh^2} = f''(x+h); \text{ and so on.}$$

Differentiating (1) then with regard to h, we have

$$f'(x+h) = \frac{df(x+h)}{dh} = A_1 + A_2 h + A_3 \frac{h^2}{2!} + A_4 \frac{h^3}{3!} + \ldots (2),$$

$$f''(x+h) = \frac{df'(x+h)}{dh} = \quad A_2 + A_3 h + A_4 \frac{h^2}{2!} + \ldots (3),$$

$$f'''(x+h) = \frac{df''(x+h)}{dh} = \quad A_3 + A_4 h + \ldots (4),$$

$$\text{etc.} = \text{etc.}$$

Putting $h=0$, we have at once from (1), (2), etc.

$$A_0 = f(x), \quad A_1 = f'(x), \quad A_2 = f''(x), \quad A_3 = f'''(x), \text{ etc.},$$

where $f'(x), f''(x), f'''(x) \ldots$ are the several differential coefficients of $f(x)$ with *respect to* x. Substituting these values in (1),

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots .$$

Ex. 1. Let $f(x) = x^n$.

Then $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, etc., and

$$f(x+h) = (x+h)^n.$$

Thus Taylor's Theorem gives the Binomial expansion

$$(x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \ldots$$

Ex. 2. Let $f(x) = \sin x$.

Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, etc., and

$$f(x+h) = \sin(x+h).$$

Thus we obtain

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + .$$

EXAMPLES.

Prove the following results :

1. $e^{x+h} = e^x + he^x + \dfrac{h^2}{2!}e^x + \dfrac{h^3}{3!}e^x + \ldots\ldots$.

2. $\tan^{-1}(x+h)$
$$= \tan^{-1}x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} - \frac{1-3x^2}{(1+x^2)^3}\frac{h^3}{3} + \ldots .$$

3. $\sin^{-1}(x+h)$
$$= \sin^{-1}x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{\frac{3}{2}}}\frac{h^2}{2!} + \frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}\frac{h^3}{3!} + \ldots .$$

4. $\sec^{-1}(x+h)$
$$= \sec^{-1}x + \frac{h}{x\sqrt{x^2-1}} - \frac{2x^2-1}{x^2(x^2-1)^{\frac{3}{2}}}\frac{h^2}{2!} + \ldots .$$

5. $\log\sin(x+h)$
$$= \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \ldots .$$

71. Stirling's or Maclaurin's Theorem.

If in Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

we put 0 for x, and x for h, we arrive at the result

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$
$$+ \frac{x^n}{n!}f^n(0) + \ldots$$

the meaning of $f^r(0)$ being that $f(x)$ is to be differentiated r times with respect to x, and then x is to be put zero in the result.

This result is generally known as Maclaurin's Theorem. Being a form of Taylor's Theorem it is subject to similar limitations.

Ex. 1. Expand $\sin x$ in powers of x.

Here
$$\begin{aligned} f(x) &= \sin x, & \text{Hence } f(0) &= 0,\\ f'(x) &= \cos x, & f'(0) &= 1,\\ f''(x) &= -\sin x, & f''(0) &= 0,\\ f'''(x) &= -\cos x, & f'''(0) &= -1,\\ &\&c. & &\&c.\\ f^n(x) &= \sin\left(x+\frac{n\pi}{2}\right). & f^n(0) &= \sin\frac{n\pi}{2} \end{aligned}$$

Thus
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \sin\frac{n\pi}{2} + \dots$$

Ex. 2. Expand $\log\cos x$ in powers of x.

Here
$$\begin{aligned} f(x) &= \log\cos x,\\ f'(x) &= -\tan x = -t, \text{ say},\\ f''(x) &= -\sec^2 x = -(1+t^2),\\ f'''(x) &= -2\tan x\sec^2 x = -2t(1+t^2),\\ f^{(4)}(x) &= -2(1+3t^2)(1+t^2) = -2(1+4t^2+3t^4),\\ f^{(5)}(x) &= -2(8t+12t^3)(1+t^2) = -2(8t+20t^3+12t^5),\\ f^{(6)}(x) &= -2(8+60t^2+60t^4)(1+t^2) = -2(8+68t^2+120t^4+60t^6),\\ &\text{etc.} \end{aligned}$$

Whence
$$f(0) = \log\cos 0 = \log 1 = 0,$$
and
$$f'(0) = f^{(3)}(0) = f^{(5)}(0) = \dots = 0,$$
also
$$f''(0) = -1,\quad f^{(4)}(0) = -2,\quad f^{(6)}(0) = -16, \text{ etc.}$$

Hence
$$\log\cos x = -\frac{x^2}{2!} - 2\frac{x^4}{4!} - 16\frac{x^6}{6!} - \text{etc.}$$

EXAMPLES.

Apply Maclaurin's Theorem to prove

1. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \qquad + \dots .$

2. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + \dots .$

3. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \dots .$

4. $e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \dots .$

5. $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{x^4}{192} \dots .$

6. $e^{ax}\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \dots$

$$+ \frac{(a^2+b^2)^{\frac{n}{2}}}{n!}x^n \cos\left(n \tan^{-1}\frac{b}{a}\right) + \dots .$$

72. METHOD III. **By the formation of a Differential Equation.** First form a differential equation as in Ex. 3, Art. 60, etc., and assume the series

$$a_0 + a_1x + a_2x^2 + \dots$$

for the expansion.

Substitute the series for y in the differential equation and equate coefficients of like powers of x in the resulting identity. We thus obtain sufficient equations to find all the coefficients except one or two of the first which may easily be obtained from the values of $f(0)$, $f'(0)$, etc.

Ex. 1. **To apply this method to the expansion of $(1+x)^n$.**

Let $\quad y = (1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \dots \dots (1).$

Then $\quad y_1 = n(1+x)^{n-1}$ or $(1+x)y_1 = ny \dots \dots \dots (2).$

But $\quad y_1 = a_1 + 2a_2x + 3a_3x^2 + \dots \dots \dots (3).$

Therefore substituting from (1) and (3) in the differential equation (2)

$$(1+x)(a_1+2a_2x+3a_3x^2+\dots)=n(a_0+a_1x+a_2x^2+\dots).$$

Hence, comparing coefficients

$$a_1=na_0,$$
$$2a_2+a_1=na_1,$$
$$3a_3+2a_2=na_2, \quad \text{etc.,}$$

and by putting $\quad x=0$ **in equation (1),**

giving
$$a_1=n,$$
$$a_2=\frac{n-1}{2}a_1=\frac{n(n-1)}{2!},$$
$$a_3=\frac{n-2}{3}a_2=\frac{n(n-1)(n-2)}{3!}, \quad \text{etc.,}$$
$$a_r=\frac{n-r+1}{r}a_{r-1}=\frac{n(n-1)\dots(n-r+1)}{r!},$$

whence
$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

Ex. 2. Let
$$y=f(x)=(\sin^{-1}x)^2.$$
$$y_1=2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}},$$
$$\therefore\quad (1-x^2)\,y_1^2=4y.$$

Differentiating, and dividing by $2y_1$, we have

$$(1-x^2)\,y_2=xy_1+2 \quad \dots\dots \quad (1).$$

Now, let $\quad y=a_0+a_1x+a_2x^2+\dots+a_nx^n+a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\dots,$

therefore

$$y_1=a_1+2a_2x+\dots+na_nx^{n-1}+(n+1)a_{n+1}x^n+(n+2)a_{n+2}x^{n+1}+\dots,$$

and

$$y_2=2a_2+\dots+n(n-1)a_nx^{n-2}+(n+1)na_{n+1}x^{n-1}+(n+2)(n+1)a_{n+2}x^n+\dots.$$

Picking out the coefficient of x^n in the equation (*which may be done without actual substitution*) we have

$$(n+2)(n+1)a_{n+2}-n(n-1)a_n=na_n;$$

therefore
$$a_{n+2}=\frac{n^2}{(n+1)(n+2)}a_n \quad \dots\dots\dots \quad (2).$$

Now, $$a_0 = f(0) = (\sin^{-1} 0)^2,$$
and if we consider $\sin^{-1} x$ to be the *smallest positive angle* whose sine is x,
$$\sin^{-1} 0 = 0.$$

Hence $$a_0 = 0.$$

Again, $$a_1 = f'(0) = 2 \sin^{-1} 0 \cdot \frac{1}{\sqrt{1-0}} = 0,$$

and $$a_2 = \tfrac{1}{2} f''(0) = \tfrac{1}{2}\left(\frac{2}{1-0} + 0\right) = 1.$$

Hence, from equation (2), a_3, a_5, a_7, .., are each $= 0$,

and $$a_4 = \frac{2^2}{3 \cdot 4} \cdot a_2 = \frac{2^2}{3 \cdot 4} = \frac{2^2}{4!} 2,$$
$$a_6 = \frac{4^2}{5 \cdot 6} \cdot a_4 = \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} = \frac{2^2 \cdot 4^2}{6!} \cdot 2,$$
etc. = etc.;

therefore
$$(\sin^{-1} x)^2 = \frac{2x^2}{2!} + \frac{2^2}{4!} 2x^4 + \frac{2^2 \cdot 4^2}{6!} 2x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{8!} 2x^8 + \ldots$$

A different method of proceeding is indicated in the following example :—

Ex. 3. Let
$$y = \sin(m \sin^{-1} x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots \qquad (1).$$

Then $$y_1 = \cos(m \sin^{-1} x) \frac{m}{\sqrt{1-x^2}},$$

whence $$(1-x^2) y_1^2 = m^2 (1-y^2).$$

Differentiating again, and dividing by $2y_1$, we have
$$(1-x^2) y_2 - x y_1 + m^2 y = 0 \ \ldots\ldots\ldots\ldots (2).$$

Differentiating this n times by Leibnitz's Theorem
$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0. \ \ldots\ldots\ldots (3).$$

Now $$a_0 = (y)_{x=0} = \sin(m \sin^{-1} 0) = 0,$$
(assuming that $\sin^{-1} x$ is the smallest positive angle whose sine is x)
$$a_1 = (y_1)_{x=0} = m,$$
$$a_2 = (y_2)_{x=0} = 0,$$
etc.
$$a_n = (y_n)_{x=0}.$$

Hence, putting $x=0$ in equation (3),

$$a_{n+2}=-(m^2-n^2)\,a_n.$$

Hence $a_4, a_6, a_8, \ldots$, each $=0$,

and
$$a_3=-(m^2-1^2)\,a_1=-m\,(m^2-1^2),$$
$$a_5=-(m^2-3^2)\,a_3=m\,(m^2-1^2)\,(m^2-3^2),$$
$$a_7=-(m^2-5^2)\,a_5=-m\,(m^2-1^2)\,(m^2-3^2)\,(m^2-5^2),$$
etc.

Whence

$$\sin(m\sin^{-1}x)=mx-\frac{m\,(m^2-1^2)}{3\,!}x^3+\frac{m\,(m^2-1^2)\,(m^2-3^2)}{5\,!}x^5$$
$$-\frac{m\,(m^2-1^2)\,(m^2-3^2)\,(m^2-5^2)}{7!}x^7+\ldots.$$

The corresponding series for $\cos(m\sin^{-1}x)$ is

$$\cos(m\sin^{-1}x)=1-\frac{m^2x^2}{2\,!}+\frac{m^2\,(m^2-2^2)}{4\,!}x^4$$
$$-\frac{m^2\,(m^2-2^2)\,(m^2-4^2)}{6\,!}x^6+\ldots.$$

If we write $x=\sin\theta$ these series become

$$\sin m\theta=m\sin\theta-\frac{m\,(m^2-1^2)}{3\,!}\sin^3\theta$$
$$+\frac{m\,(m^2-1^2)\,(m^2-3^2)}{5\,!}\sin^5\theta-\text{etc.},$$

$$\cos m\theta=1-\frac{m^2}{2\,!}\sin^2\theta+\frac{m^2\,(m^2-2^2)}{4\,!}\sin^4\theta$$
$$-\frac{m^2\,(m^2-2^2)\,(m^2-4^2)}{6\,!}\sin^6\theta+\text{etc.}$$

EXAMPLES.

1. Apply this method to find the known expansions of

$$a^x,\quad \log(1+x),\quad \sin x,\quad \tan^{-1}x.$$

2. If $\quad y=\sin^{-1}x=a_0+a_1x+a_2x^2+a_3x^3+\ldots,$

prove (1) $(1-x^2)\,y_2=xy_1$,

(2) $(n+1)(n+2)a_{n+2}=n^2a_n$,

(3) $\sin^{-1}x=x+\dfrac{1}{2}\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\ldots.$

3. If $y=e^{a\sin^{-1}x}=a_0+a_1x+a_2x^2+a_3x^3+\dots,$

prove (1) $(1-x^2)\,y_2=xy_1+a^2y,$

(2) $(n+1)(n+2)\,a_{n+2}=(n^2+a^2)\,a_n,$

(3) $$e^{a\sin^{-1}x}=1+ax+\frac{a^2x^2}{2\,!}+\frac{a(a^2+1)}{3\,!}x^3+\frac{a^2(a^2+2^2)}{4\,!}x^4 + \frac{a(a^2+1)(a^2+3^2)}{5\,!}x^5+\dots,$$

(4) Deduce from (3) by expanding the left side by the exponential theorem and equating coefficients of a, a^2, a^3... the series for $\sin^{-1}x$, $(\sin^{-1}x)^2$, $(\sin^{-1}x)^3$.

4. Prove that

$$\frac{(\tan^{-1}x)^2}{2\,!}=\frac{x^2}{2}-\left(1+\frac{1}{3}\right)\frac{x^4}{4}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6}-\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)\frac{x^8}{8}+\dots.$$

5. Prove that

(a) $$\frac{1}{2}[\log(x+\sqrt{1+x^2})]^2=\frac{x^2}{2}-\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2\,.\,4}{3\,.\,5}\cdot\frac{x^6}{6}-\dots,$$

(b) $$\frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}=x-\frac{2}{3}x^3+\frac{2\,.\,4}{3\,.\,5}\cdot x^5-\dots.$$

73. METHOD IV. **Differentiation or integration of a known series.** The method of treatment is best indicated by examples.

Ex. 1. If we differentiate the series

$$\sin^{-1}x=x+\frac{1}{2}\,\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\,\frac{x^5}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,\frac{x^7}{7}+$$

we obtain the binomial expansion

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,5}x^6+\dots,$$

and it is clear that we must be able by a reverse process (integration) to infer the first series from the second.

The student unacquainted with integration may obtain the expansion of $\sin^{-1}x$ from that of $(1-x^2)^{-\frac{1}{2}}$ as follows:

Let $\sin^{-1}x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots,$
then differentiating

$$\frac{1}{\sqrt{1-x^2}} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + .$$

But
$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \ldots.$$

Hence $a_1 = 1,\ \ 2a_2 = 0,\ \ 3a_3 = \frac{1}{2},\ \ 4a_4 = 0,\ \ 5a_5 = \frac{1 \cdot 3}{2 \cdot 4}$, etc.

Also $a_0 = \sin^{-1} 0 = 0$
(if we take the smallest positive value of the inverse function).

Hence substituting the values of these coefficients

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \ldots\ldots.$$

Ex. 2. We have proved in Ex. 2 Art. 72 that

$$\frac{(\sin^{-1}x)^2}{2!} = \frac{x^2}{2!} + \frac{2^2x^4}{4!} + \frac{2^2 \cdot 4^2}{6!}x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{8!}x^8 + \ldots.$$

Hence differentiating we arrive at a new series

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2^2}{3!}x^3 + \frac{2^2 \cdot 4^2}{5!}x^5 + \frac{2^2 \cdot 4^2 \cdot 6^2}{7!}x^7 + \ldots.$$

If we put $x = \sin\theta$ we may write this as

$$\frac{2\theta}{\sin 2\theta} = 1 + \frac{2^2}{3!}\sin^2\theta + \frac{2^2 \cdot 4^2}{5!}\sin^4\theta + \frac{2^2 \cdot 4^2 \cdot 6^2}{7!}\sin^6\theta + \ldots,$$

or
$$= 1 + \frac{2}{3}\sin^2\theta + \frac{2 \cdot 4}{3 \cdot 5}\sin^4\theta + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\sin^6\theta + \ldots.$$

EXAMPLES.

1. Obtain in this manner the expansion of

$$\log(1+x), \quad \tan^{-1}x, \quad \log\frac{1+x}{1-x}.$$

2. Prove

$$\log(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \ldots.$$

3. Expand

$$\sin^{-1}\frac{2x}{1+x^2}, \quad \tan^{-1}\frac{x}{\sqrt{1-x^2}}, \quad \tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

in powers of x.

4. Prove

$$\left(\frac{\theta}{\sin\theta}\right)^2$$

$$=1+\frac{2^2}{3\,.\,4}\sin^2\theta+\frac{2^2\,.\,4^2}{3\,.\,4\,.\,5\,.\,6}\sin^4\theta+\frac{2^2\,.\,4^2\,.\,6^2}{3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8}\sin^6\theta+\ldots.$$

5. Prove that

(1) $$\frac{e^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$=1+\frac{ax}{1\,!}+\frac{(a^2+1^2)x^2}{2\,!}+\frac{a(a^2+2^2)x^3}{3\,!}+\frac{(a^2+1^2)(a^2+3^2)x^4}{4\,!}+\ldots,$$

(2) $$\frac{\quad}{\cos\theta}$$

$$=1+\frac{\sin\theta}{1\,!}+\frac{(1+1^2)\sin^2\theta}{2\,!}+\frac{(1+2^2)\sin^3\theta}{3\,!}$$

$$+\frac{(1+1^2)(1+3^2)\sin^4\theta}{4\,!}+\ldots.$$

EXAMPLES.

1. Prove

$$\log(1+\tan x)=x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots.$$

2. Prove

$$e^{x\cos x}=1+x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{11x^4}{24}-\frac{x^5}{5}\ldots.$$

3. Prove

$$\log\left\{\frac{1}{x}e^{\frac{x}{2}}\log(1+x)\right\}=\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4\ldots.$$

4. Prove

$$\log(1-x+x^2)=-x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{3}-\frac{x^7}{7}+\frac{x^8}{8}\ldots.$$

5. Prove $\cosh(x\cos x) = 1 + \frac{x^2}{2} - \frac{11x^4}{24} \dots,$

$$\sinh(x\cos x) = x - \frac{x^3}{3} - \frac{x^5}{5} \dots.$$

6. Prove $\log\frac{\tan x}{x} = \frac{x^2}{3} + \frac{7}{90}x^4 \dots$

7. Prove

$$\cos^{-1}(\tanh\log x) = \pi - 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right\}.$$

8. Prove

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \frac{x}{2} - \frac{x^3}{6} + \frac{x^5}{10} - \frac{x^7}{14} + \dots.$$

9. Prove $\log(3x + 4x^3 + \sqrt{1 + 9x^2 + 24x^4 + 16x^6})$

$$= 3\left\{x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \dots\right\}.$$

10. Prove that

(a) $(1-x^2)^{\frac{1}{2}}\sin^{-1}x = x - \frac{x^3}{3} - \frac{2}{3}\frac{x^5}{5} - \frac{2.4}{3.5}\cdot\frac{x^7}{7} - \dots,$

(b) $\theta\cot\theta = 1 - \frac{\sin^2\theta}{3} - \frac{2}{3}\frac{\sin^4\theta}{5} - \frac{2.4}{3.5}\frac{\sin^6\theta}{7} - \dots,$

(c) $\frac{\pi}{4} = 1 - \frac{1}{3}\left(\frac{1}{2}\right) - \frac{2}{3}\frac{1}{5}\left(\frac{1}{2}\right)^2 - \frac{2.4}{3.5}\cdot\frac{1}{7}\left(\frac{1}{2}\right)^3 - .$

11. Prove that

$$(x+\sqrt{1+x^2})^n$$
$$= 1 + nx + \frac{n^2x^2}{2!} + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \dots,$$

and deduce the expansions of

$$\log(x+\sqrt{1+x^2}),\quad \frac{1}{2!}\{\log(x+\sqrt{1+x^2})\}^2,\quad \frac{1}{3!}\{\log(x+\sqrt{1+x^2})\}^3.$$

12. If $y = e^{ax}\cos bx,$

prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0,$

and hence that

$$e^{ax}\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \dots.$$

13 Prove

(a) $\sin(m\tan^{-1}x)(1+x^2)^{\frac{m}{2}}$

$$= mx - \frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)(m-2)(m-3)(m-4)}{5!}x^5 - \dots,$$

(b) $\cos(m\tan^{-1}x)(1+x^2)^{\frac{m}{2}}$

$$= 1 - \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)(m-3)}{4!}x^4 - \dots.$$

14. Deduce from 13 (a)

$\tan^{-1}x \log\sqrt{1+x^2}$

$$= \left(\frac{1}{1}+\frac{1}{2}\right)\frac{x^3}{3} - \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^5}{5} + \left(\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{6}\right)\frac{x^7}{7} - \dots$$

15 Prove

(a) $$\frac{\cosh\theta}{\cos\theta} = 1 + \frac{1^2+1^2}{2!}\sin^2\theta + \frac{(1^2+1^2)(1^2+3^2)}{4!}\sin^4\theta + \dots,$$

(b) $$\frac{\sinh\theta}{\cos\theta} = \frac{1^2}{1!}\sin\theta + \frac{(1^2+2^2)}{3!}\sin^3\theta + \frac{(1^2+2^2)(1^2+4^2)}{5!}\sin^5\theta + \dots$$

16 Prove

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta\sin\theta - \frac{(h\sin\theta)^2}{2}\sin 2\theta$$
$$+ \frac{(h\sin\theta)^3}{3}\sin 3\theta - \frac{(h\sin\theta)^4}{4}\sin 4\theta + \text{etc.},$$

where $x = \cot\theta$.

17. Deduce from Ex. 16

(a) $$\frac{\pi}{2} - \theta = \cos\theta\sin\theta + \frac{\cos^2\theta}{2}\sin 2\theta + \frac{\cos^3\theta}{3}\sin 3\theta + \dots,$$

by putting $h = -x = -\cot\theta$.

(b) $$\frac{\pi-\theta}{2} = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \frac{1}{4}\sin 4\theta + \dots,$$

by putting $h = -\sqrt{1+x^2}$.

(c) $$\frac{\pi}{2} = \frac{\sin\theta}{\cos\theta} + \frac{1}{2}\frac{\sin 2\theta}{\cos^2\theta} + \frac{1}{3}\frac{\sin 3\theta}{\cos^3\theta} + \frac{1}{4}\frac{\sin 4\theta}{\cos^4\theta} + \dots$$

by putting $h = -x - x^{-1}$.

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18. Show that

(*a*) $\dfrac{1}{2\,!}\dfrac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}$

$$=\frac{x^2}{2\,!}+1^2\,.\,3^2\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\frac{x^4}{4\,!}+1^2\,.\,3^2\,.\,5^2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{x^6}{6\,!}+\dots,$$

(*b*) $\dfrac{\theta^2}{\sin 2\theta}$

$$=\frac{\sin\theta}{2\,!}+1^2\,.\,3^2\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\frac{\sin^3\theta}{4\,!}+1^2\,.\,3^2\,.\,5^2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{\sin^5\theta}{6\,!}+\dots.$$

19. Prove

$$\frac{(\tan^{-1}x)^3}{3\,!}=\frac{1}{2}\frac{x^3}{3}-\left\{\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)\right\}\frac{x^5}{5}+\left\{\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right\}\frac{x^7}{7}-\dots.$$

20. Prove

(*a*) $$\frac{\text{vers}^{-1}x}{\sqrt{2x}}=1+\frac{1}{3\,.\,4}x+\frac{1\,.\,3}{5\,.\,4^2}\frac{x^2}{2\,!}+\frac{1\,.\,3\,.\,5}{7\,.\,4^3}\frac{x^3}{3\,!}+\dots,$$

(*b*) $$\frac{(\text{vers}^{-1}x)^2}{2}=x+\frac{1}{3}\frac{x^2}{2}+\frac{1\,.\,2}{3\,.\,5}\cdot\frac{x^3}{3}+\frac{1\,.\,2\,.\,3}{3\,.\,5\,.\,7}\frac{x^4}{4}+\dots.$$

21. Prove that

$$\frac{f(x+h)+f(x-h)}{2}=f(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^4}{4\,!}f''''(x)+\dots.$$

22. Prove that

(*a*) $f(mx)$

$$f(x)+(m-1)\,x f'(x)+(m-1)^2\frac{x^2}{2\,!}f''(x)+(m-1)^3\frac{x^3}{3\,!}f'''(x)+\dots,$$

(*b*) $f\left(\dfrac{x^2}{1+x}\right)$

$$=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x)^2}\frac{1}{2\,!}f''(x)-\frac{x^3}{(1+x)^3}\frac{1}{3\,!}f'''(x)+\dots,$$

(*c*) $f(x)=f(0)+xf'(x)-\dfrac{x^2}{2\,!}f''(x)+\dfrac{x^3}{3\,!}f'''(x)-\text{etc.}$

CHAPTER VII.

INFINITESIMALS.

74. Orders of Smallness.

If we conceive any magnitude A divided into any large number of equal parts, say a billion (10^{12}), then each part $\frac{A}{10^{12}}$ is extremely small, and for all practical purposes negligible, in comparison with A. If this part be again subdivided into a billion equal parts, each $= \frac{A}{10^{24}}$, each of these last is extremely small in comparison with $\frac{A}{10^{12}}$, and so on. We thus obtain a series of magnitudes, A, $\frac{A}{10^{12}}$, $\frac{A}{10^{24}}$, $\frac{A}{10^{36}}$, ..., each of which is excessively small in comparison with the one which precedes it, but very large compared with the one which follows it. This furnishes us with what we may designate *a scale of smallness*.

75. More generally, if we agree to consider any given fraction f as being small in comparison with unity, then fA will be small in comparison with A, and we may term the expressions fA, f^2A, f^3A, ..., *small quantities of the first, second, third, etc., orders*; and the numerical quantities f, f^2, f^3, ..., may be called *small fractions* of the first, second, third, etc., orders.

Thus, supposing A to be any given finite magnitude, any given fraction of A is at our choice to designate a small quantity of the first order in comparison with A. When this is chosen, any quantity which has to this small quantity of the first order a ratio which is a small fraction of the first order, is itself a small quantity of the second order. Similarly, any quantity whose ratio to a small quantity of the second order is a small fraction of the first order is a small quantity of the third order, and so on. So that generally, if a small quantity be such that its ratio to a small quantity of the p^{th} order be a small fraction of the q^{th} order, it is itself termed a small quantity of the $(p+q)^{\text{th}}$ order.

76. Infinitesimals.

If these small quantities Af, Af^2, Af^3, ..., be all quantities whose limits are zero, then supposing *f made smaller than any assignable quantity* by sufficiently increasing its denominator, these small quantities of the first, second, third, etc., orders are termed *infinitesimals of the first, second, third, etc., orders.*

From the nature of an infinitesimal it is clear that, *if any equation contain finite quantities and infinitesimals, the infinitesimals may be rejected.*

77. PROP. *In any equation between infinitesimals of different orders, none but those of the lowest order need be retained.*

Suppose, for instance the equation to be

$$A_1 + B_1 + C_1 + D_2 + E_2 + F_3 + \ldots = 0 \ldots\ldots \text{(i)},$$

each letter denoting an infinitesimal of the order indicated by the suffix.

Then, dividing by A_1,

$$1 + \frac{B_1}{A_1} + \frac{C_1}{A_1} + \frac{D_2}{A_1} + \frac{E_2}{A_1} + \frac{F_3}{A_1} + \ldots = 0 \ldots\ldots \text{(ii)},$$

the limiting ratios $\frac{B_1}{A_1}$ and $\frac{C_1}{A_1}$ are finite, while $\frac{D_2}{A_1}$, $\frac{E_2}{A_1}$, are infinitesimals of the first order, $\frac{F_3}{A_1}$ is an infinitesimal of the second order, and so on. Hence, by Art. 76, equation (ii) may be replaced by

$$1+\frac{B_1}{A_1}+\frac{C_1}{A_1}=0,$$

and therefore equation (i) by

$$A_1+B_1+C_1=0,$$

which proves the statement.

78. Prop. *In any equation connecting infinitesimals we may substitute for any one of the quantities involved any other which differs from it by a quantity of higher order.*

For if $A_1+B_1+C_1+D_2+\ldots=0$ be the equation, and if $A_1=F_1+f_2$, f_2 denoting an infinitesimal of higher order than F_1, we have $F_1+B_1+C_1+f_2+D_2+\ldots=0$,

i.e. by the last proposition we may write

$$F_1+B_1+C_1=0,$$

which may therefore, if desirable, replace the equation

$$A_1+B_1+C_1=0.$$

79. Illustrations.

(1) Since
$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$$
and
$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$$

$\sin\theta$, $1-\cos\theta$, $\theta-\sin\theta$ are respectively of the first, second, and third orders of small quantities, when θ is of the first order; also, 1 may be written instead of $\cos\theta$ if second order quantities are to be rejected, and θ for $\sin\theta$ when cubes and higher powers are rejected.

(2) Again, suppose AP the arc of a circle of centre O and radius a. Suppose the angle $AOP\,(=\theta)$ to be a small quantity of the first order. Let PN be the perpendicular from P upon OA and AQ the tangent at A, meeting OP produced in Q. Join P, A.

Then arc $AP=a\theta$ and is of the first order,

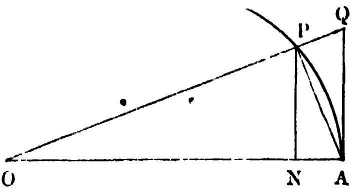
$NP=a\sin\theta$ do. do.,

$AQ=a\tan\theta$ do. do.,

chord $AP=2a\sin\frac{\theta}{2}$ do. do.,

$NA=a\,(1-\cos\theta)$ and is of the second order.

So that $OP-ON$ is a small quantity of the second order.



Again, arc AP − chord $AP=a\theta-2a\sin\frac{\theta}{2}$

$$=a\theta-2a\left(\frac{\theta}{2}-\frac{\theta^3}{8\,.\,3!}+\dots\right)$$

$$=\frac{a\theta^3}{4\,.\,3!}-\text{etc.},$$

and is of the third order.

$$PQ-NA=NA\,(\sec\theta-1)$$

$$=NA\,.\,\frac{2\sin^2\frac{\theta}{2}}{\cos\theta}$$

= (second order)(second order)

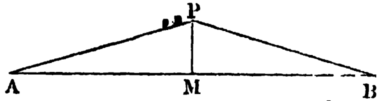
= fourth order of small quantities,

and similarly for others.

80. *The base angles of a triangle being given to be small quantities of the first order, to find the order of the difference between the base and the sum of the sides.*

By what has gone before, (Art. 79 (2)), if APB be the

triangle and PM the perpendicular on AB, $AP - AM$



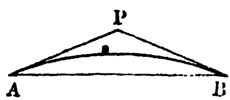
and $BP - BM$ are both small quantities of the second order as compared with AB.

Hence $AP + PB - AB$ is of the second order compared with AB.

If AB itself be of the first order of small quantities, then $AP + PB - AB$ is *of the third order*.

81. *Degree of approximation in taking a small chord for a small arc in any curve.*

Let AB be an arc of a curve supposed continuous between A and B, and so small as to be concave at each



point throughout its length to the foot of the perpendicular from that point upon the chord. Let AP, BP be the tangents at A and B. Then, when A and B are taken sufficiently near together, the chord AB and the angles at A and B may each be considered small quantities of at least the first order, and therefore, by what has gone before, $AP + PB - AB$ will be at least of the third order. Now we may take *as an axiom* that the *length of the arc AB is intermediate between the length of the chord AB and the sum of the tangents AP, BP.* Hence the difference of the arc AB and the chord AB, which is less than that between $AP + PB$ and the chord AB, must be at least *of the third order.*

EXAMPLES.

1. In the figure on page 86 suppose PM drawn at right angles to AQ, and prove

(*a*) Segment cut off by AP is of the third order of small quantities,

(*b*) Triangle PNA is of the third order,

(*c*) Triangle PQM is of the fifth order.

2. OA_1B is a triangle right-angled at A_1 and of which the angle at O is small and of the first order. A_1B_1 is drawn perpendicular to OB, B_1A_2 to A_1B, A_2B_2 to OB, and so on

Prove

(*a*) A_nB_n is a small quantity of the $(2n-1)^{\text{th}}$ order,

(*b*) B_nA_{n+1} is of the $2n^{\text{th}}$ order,

(*c*) B_nB is of the $2n^{\text{th}}$ order,

(*d*) triangle BA_mB_n is of the $(2m+2n-1)^{\text{th}}$ order.

3. A straight line of constant length slides between two straight lines at right angles, viz. CAa, CbB; AB, ab are two positions of the line, and P their point of intersection. Show that, in the limit, when the two positions coincide, we have

$$\frac{Aa}{Bb}=\frac{CB}{CA} \text{ and } \frac{PA}{PB}=\frac{CB^2}{CA^2}.$$

4. From a point T in a radius of a circle, produced, a tangent TP is drawn to the circle touching it in P. PN is drawn perpendicular to the radius OA. Show that, in the limit when P moves up to A,

$$NA=AT.$$

5. Tangents are drawn to a circular arc at its middle point and at its extremities; show that the area of the triangle formed by the chord of the arc and the two tangents at the extremities is ultimately four times that of the triangle formed by the three tangents.

6. A regular polygon of n sides is inscribed in a circle. Show that when n is very great the ratio of the difference of the circumferences to the circumference of the circle is approximately

$$\pi^2/6n^2.$$

7. Show that the difference between the perimeters of the earth and that of an inscribed regular polygon of ten thousand sides is less than a yard (rad. of Earth $=4000$ miles).

8. The sides of a triangle are 5 and 6 feet and the included angle exceeds 60° by $10''$. Calculating the third side for an angle of 60°, find the correction to be applied for the extra $10''$.

9. A person at a distance q from a tower of height p observes that a flag-pole upon the top of it subtends an angle θ at his eye. Neglecting his height, show that if the observed angle be subject to a small error α, the corresponding error in the length of the pole has to the calculated length the ratio

$$q\alpha \operatorname{cosec} \theta / (q \cos \theta - p \sin \theta).$$

10. If in the equation $\sin(\omega - \theta) = \sin \omega \cos \alpha$, θ be small, show that its approximate value is

$$2 \tan \omega \sin^2 \frac{\alpha}{2} \left(1 - \tan^2 \omega \sin^2 \frac{\alpha}{2}\right.$$

[I. C. S.]

11. A small error x is made in measuring the side a of a triangle, a small error y in measuring b, and a small error n'' in measuring C. Prove that the consequent errors in A and B are each $\frac{1}{2}n''$, provided the relation

$$2 \frac{bx - ay}{a^2 - b^2} \sin C = n \sin 1''$$

be satisfied. [I. C. S., 1892.]

CHAPTER VIII.

TANGENTS AND NORMALS.

82. Equation of TANGENT.

It was shown in Art. 18 that the equation of the tangent at the point (x, y) on the curve $y = f(x)$ is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \text{.................(1)},$$

X and Y being the current co-ordinates of any point on the tangent.

Suppose the equation of the curve to be given in the form $f(x, y) = 0$.

It is shown in Art. 58 that

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Substituting this expression for $\frac{dy}{dx}$ in (1) we obtain

$$Y - y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(X - x),$$

or $$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0. \qquad (2)$$

for the equation of the tangent.

If the partial differential coefficients $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, etc. be denoted by f_x, f_y, etc, equation (2) may then be written

$$(X-x)f_x+(Y-y)f_y=0.$$

83. Simplification for Algebraic Curves.

If $f(x, y)$ be an algebraic function of x and y of degree n, suppose it made *homogeneous in x, y, and z by the introduction of a proper power of the linear unit z* wherever necessary. Call the function thus altered $f(x, y, z)$. Then $f(x, y, z)$ is a homogeneous algebraic function of the n^{th} degree; hence we have by Euler's Theorem (Art. 59)

$$xf_x+yf_y+zf_z=nf(x, y, z)=0,$$

by virtue of the equation to the curve.

Adding this to equation (2), the equation of the tangent takes the form

$$Xf_x+Yf_y+zf_z=0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where the z is to be put $=1$ after the differentiations have been performed.

We often for the sake of symmetry write Z instead of z in this equation and write the tangent in the form

$$Xf_x+Yf_y+Zf_z=0.$$

Ex. $$f(x, y)\equiv x^4+a^2xy+b^3y+c^4=0.$$

The equation, when made *homogeneous in x, y, z by the introduction of a proper power of z, is*

$$f(x, y, z)\equiv x^4+a^2xyz^2+b^3yz^3+c^4z^4=0,$$

and $$f_x=4x^3+a^2yz^2,$$

$$f_y=a^2xz^2+b^3z^3,$$

$$f_z=2a^2xyz+3b^3yz^2+4c^4z^3.$$

Substituting these in Equation 3, and putting $Z = z = 1$, we have for the equation of the tangent to the curve at the point (x, y)

$$X(4x^3 + a^2y) + Y(a^2x + b^3) + 2a^2xy + 3b^3y + 4c^4 = 0.$$

With very little practice the introduction of the z can be performed *mentally*. It is generally *more advantageous* to use equation (3) than equation (2), because (3) gives the result *in its simplest form*, whereas if (2) be used it is often necessary to reduce by substitutions from the equation of the curve.

84. NORMAL.

DEF. *The normal at any point of a curve is a straight line through that point and perpendicular to the tangent to the curve at that point.*

Let the axes be assumed rectangular. The equation of the normal may then be at once written down. For if the equation of the curve be

$$y = f(x),$$

the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x),$$

and the normal is therefore

$$(X - x) + (Y - y)\frac{dy}{dx} = 0.$$

If the equation of the curve be given in the form

$$f(x, y) = 0,$$

the equation of the tangent is

$$(X - x)f_x + (Y - y)f_y = 0,$$

and therefore that of the normal is

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y}.$$

Ex. 1. Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

This requires z^2 in the last term to make a homogeneous equation in x, y, and z. We have then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0.$$

Hence the equation of the tangent is

$$X\,.\,\frac{2x}{a^2}+Y\,.\,\frac{2y}{b^2}-z\,.\,2z=0,$$

where z is to be put $=1$. Hence we get

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1 \text{ for the tangent,}$$

and therefore
$$\frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}} \text{ for the normal.}$$

Ex. 2. Take the general equation of a conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

When made homogeneous this becomes

$$ax^2+2hxy+by^2+2gxz+2fyz+cz^2=0.$$

The equation of the tangent is therefore

$$X(ax+hy+g)+Y(hx+by+f)+gx+fy+c=0,$$

and that of the normal is

$$\frac{X-x}{ax+hy+g}=\frac{Y-y}{hx+by+f}.$$

Ex. 3. Consider the curve $\frac{y}{a}=\log\sec\frac{x}{a}$.

Then
$$\frac{dy}{dx}=\tan\frac{x}{a},$$

and the equation of the tangent is

$$Y-y=\tan\frac{x}{a}(X-x),$$

and of the normal

$$(Y-y)\tan\frac{x}{a}+(X-x)=0.$$

85. If $f(x, y) = 0$ and $F(x, y) = 0$ be two curves intersecting at the point x, y, their respective tangents at that point are

$$Xf_x + Yf_y + Zf_z = 0,$$

and $$XF_x + YF_y + ZF_z = 0.$$

The angle at which these lines cut is

$$\tan^{-1}\frac{f_xF_y - f_yF_x}{f_xF_x + f_yF_y}.$$

Hence if the curves touch

$$f_x/F_x = f_y/F_y;$$

and if they cut orthogonally,

$$f_xF_x + f_yF_y = 0.$$

Ex. Find the angle of intersection of the curves

$$x^3 - 3xy^2 = a,$$
$$3x^2y - y^3 = b.$$

Calling the left-hand members f and F respectively, we have

$$f_x = 3(x^2 - y^2) = F_y,$$
$$f_y = -6xy = -F_x.$$

Hence clearly $$f_xF_x + f_yF_y = 0,$$

and the curves cut orthogonally.

86. If the form of a curve be given by the equations

$$x = \phi(t), \quad y = \psi(t)$$

the tangent at the point determined by the third variable t is by equation 1, Art. 82,

$$Y - \psi(t) = \frac{\psi'(t)}{\phi'(t)}\{X - \phi(t)\},$$

or $$X\psi'(t) - Y\phi'(t) = \phi(t)\psi'(t) - \psi(t)\phi'(t).$$

Similarly by Art. 84 the corresponding normal is

$$X\phi'(t) + Y\psi'(t) = \phi(t)\phi'(t) + \psi(t)\psi'(t).$$

EXAMPLES.

1. Find the equations of the tangents and normals at the point (x, y) on each of the following curves:—

(1) $x^2+y^2=c^2$.
(2) $y^2=4ax$.
(3) $xy=k^2$.
(4) $y=c\cosh\dfrac{x}{c}$.
(5) $x^2y+xy^2=a^3$.
(6) $e^y=\sin x$.
(7) $x^3-3axy+y^3=0$.
(8) $(x^2+y^2)^2=a^2(x^2-y^2)$.

2. Write down the equations of the tangents and normals to the curve $y(x^2+a^2)=ax^2$ at the points where $y=\dfrac{a}{4}$.

3. Prove that $\dfrac{x}{a}+\dfrac{y}{b}=1$ touches the curve $y=be^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y.

4. Find where the tangent is parallel to the axis of x and where it is perpendicular to that axis for the following curves:—

(a) $ax^2+2hxy+by^2=1$.

(β) $y=\dfrac{x^3-a^3}{ax}$.

(γ) $y^3=x^2(2a-x)$.

5. Find the tangent and normal at the point determined by θ on

(a) The ellipse $\left.\begin{array}{l} x=a\cos\theta \\ y=b\sin\theta \end{array}\right\}$.

(β) The cycloid $\left.\begin{array}{l} x=a(\theta+\sin\theta) \\ y=a(1-\cos\theta) \end{array}\right\}$.

(γ) The epicycloid $\left.\begin{array}{l} x=A\cos\theta-B\cos\dfrac{A}{B}\theta \\ y=A\sin\theta-B\sin\dfrac{A}{B}\theta \end{array}\right\}$.

6. If $p=x\cos a+y\sin a$ touch the curve

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1,$$

prove that $$p^{\frac{m}{m-1}}=(a\cos a)^{\frac{m}{m-1}}+(b\sin a)^{\frac{m}{m-1}}.$$

Hence write down the polar equation of the locus of the foot of the perpendicular from the origin on the tangent to this curve.

Examine the cases of an ellipse and of a rectangular hyperbola.

7. Find the condition that the conics

$$ax^2+by^2=1, \quad a'x^2+b'y^2=1$$

shall cut orthogonally.

8. Prove that, if the axes be oblique and inclined at an angle ω, the equation of the normal to $y=f(x)$ at (x, y) is

$$(Y-y)\left(\cos\omega+\frac{dy}{dx}\right)+(X-x)\left(1+\cos\omega\frac{dy}{dx}\right)=0.$$

9. Show that the parabolas $x^2=ay$ and $y^2=2ax$ intersect upon the Folium of Descartes $x^3+y^3=3axy$; and find the angles between each pair at the points of intersection.

87. Tangents at the Origin.

It will be shown in a subsequent article (124) that in the case in which a curve, whose equation is given in the rational algebraic form, passes through the origin, the equation of the tangent or tangents at that point can be at once written down by inspection; the rule being to *equate to zero the terms of lowest degree* in the equation of the curve.

Ex. 1. In the curve $x^2+y^2+ax+by=0$, $ax+by=0$ is the equation of the tangent at the origin; and in the curve $(x^2+y^2)^2=a^2(x^2-y^2)$, $x^2-y^2=0$ is the equation of a pair of tangents at the origin.

Ex. 2. Write down the equations of the tangents at the origin in the following curves: –

(α) $(x^2+y^2)^2=a^2x^2-b^2y^2.$

(β) $x^5+y^5=5ax^2y^2.$

(γ) $(y-a)^2\dfrac{x^2+y^2}{y^2}=b^2.$

GEOMETRICAL RESULTS.

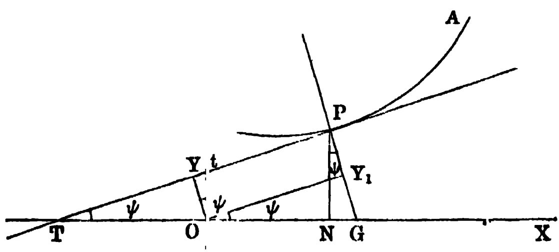
88. Cartesians. Intercepts.

From the equation $Y - y = \frac{dy}{dx}(X - x)$

it is clear that the *intercepts* which the tangent cuts off from the axes of x and y are respectively

$$x - \frac{y}{\frac{dy}{dx}} \text{ and } y - x\frac{dy}{dx},$$

for these are respectively the values of X when $Y = 0$ and of Y when $X = 0$.



Let PN, PT, PG be the ordinate, tangent, and normal to the curve, and let PT make an angle ψ with the axis of x; then $\tan\psi = \frac{dy}{dx}$. Let the tangent cut the axis of y in t, and let OY, OY_1 be perpendiculars from O, the origin, on the tangent and normal. Then the above values of the intercepts are also obvious from the figure.

89. Subtangent, etc.

DEF. The line TN is called the *subtangent* and the line NG is called the *subnormal*.

From the figure

$$Subtangent = TN = y\cot\psi = \frac{y}{\dfrac{dy}{dx}}.$$

$$Subnormal = NG = y\tan\psi = y\frac{dy}{dx}.$$

$$Normal = PG = y\sec\psi = y\sqrt{1+\tan^2\psi}$$

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)}$$

$$Tangent = TP = y\operatorname{cosec}\psi = y\frac{\sqrt{1+\tan^2\psi}}{\tan\psi}$$

$$= y\frac{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}.$$

$$OY = Ot\cos\psi = \frac{y - x\dfrac{dy}{dx}}{\sqrt{1+\tan^2\psi}} = \frac{y - x\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.$$

$$OY_1 = OG\cos\psi = \frac{ON + NG}{\sqrt{1+\tan^2\psi}} = \frac{x + y\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.$$

These and other results may of course also be obtained analytically from the equation of the tangent.

Thus if the equation of the curve be given in the form

$$f(x, y) = 0,$$

the tangent
$$Xf_x + Yf_y + Zf_z = 0$$

makes intercepts $-f_z/f_x$ and $-f_z/f_y$ upon the co-ordinate axes, and the perpendicular from the origin upon the tangent is

$$f_z/\sqrt{f_x^2 + f_y^2};$$

and indeed, any lengths or angles desired may be written down by the ordinary methods and formulae of analytical geometry.

Ex. 1. For the "chainette"

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$$

we have
$$y_1 = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Hence Subtangent $= \dfrac{y}{y_1} = c\,\dfrac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}$.

$$\text{Subnormal} = yy_1 = \frac{c}{4}\left(e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}\right).$$

$$\text{Normal} = y\sqrt{1 + y_1^2} = \frac{y^2}{c}, \text{ etc.}$$

Ex. 2. Find that curve of the class $y = \dfrac{x^n}{a^{n-1}}$ whose subnormal is constant.

Here
$$y_1 = n\frac{x^{n-1}}{a^{n-1}},$$

and
$$\text{subnormal} = yy_1 = n\frac{x^{2n-1}}{a^{2n-2}}.$$

Thus if $2n = 1$ the x disappears and leaves

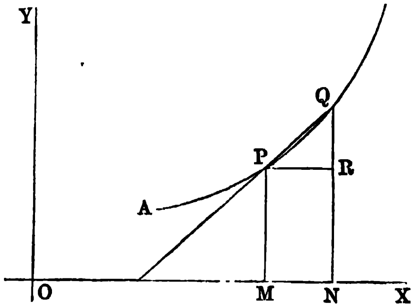
$$\text{subnormal} = \frac{a}{2},$$

and the curve is the ordinary parabola

$$y^2 = ax$$

90. Values of $\frac{ds}{dx}$, $\frac{dx}{ds}$, etc.

Let P, Q be contiguous points on a curve. Let the co-ordinates of P be (x, y) and of Q $(x + \delta x, y + \delta y)$.



Then the perpendicular $PR = \delta x$, and $RQ = \delta y$. Let the arc AP measured from some fixed point A on the curve be called s and the arc $AQ = s + \delta s$. Then arc $PQ = \delta s$. When Q travels along the curve so as to come indefinitely near to P, the arc PQ and the chord PQ differ ultimately by a quantity of higher order of smallness than the arc PQ itself. (Art. 81.)

Hence, rejecting infinitesimals of order higher than the second, we have

$$\delta s^2 = (\text{chord } PQ)^2 = (\delta x^2 + \delta y^2),$$

or
$$= Lt\left(\frac{\delta x^2}{\delta s^2} + \frac{\delta y^2}{\delta s^2}\right) = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2.$$

Similarly
$$Lt\,\frac{\delta s^2}{\delta x^2} = Lt\left(1 + \frac{\delta y^2}{\delta x^2}\right),$$

or
$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2;$$

and in the same manner
$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2.$$

If ψ be the angle which the tangent makes with the axis of x we have as in Art. 18,

$$\tan\psi = Lt\,\frac{RQ}{PR} = Lt\,\frac{\delta y}{\delta x} = \frac{dy}{dx},$$

and also

$$\cos\psi = Lt\,\frac{PR}{\text{chord } PQ} = Lt\,\frac{PR}{\text{arc } PQ} = Lt\,\frac{\delta x}{\delta s} = \frac{dx}{ds},$$

and

$$\sin\psi = Lt\,\frac{RQ}{\text{chord } PQ} = Lt\,\frac{RQ}{\text{arc } PQ} = Lt\,\frac{\delta y}{\delta s} = \frac{dy}{ds}.$$

EXAMPLES.

1. Find the length of the perpendicular from the origin on the tangent at the point x, y of the curve

$$x^4 + y^4 = c^4.$$

2. Show that in the curve $y = be^{\frac{x}{a}}$ the subtangent is of constant length.

3. Show that in the curve $by^2 = (x+a)^3$ the square of the subtangent varies as the subnormal.

4. For the parabola $y^2 = 4ax$, prove

$$\frac{ds}{dx} = \sqrt{\frac{a+x}{x}}.$$

5. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $x = a\sin\phi$

$$\frac{ds}{d\phi} = a\sqrt{1 - e^2\sin^2\phi}.$$

6. For the cycloid $\left.\begin{aligned} x &= a \operatorname{vers}\theta \\ y &= a\,(\theta + \sin\theta) \end{aligned}\right\}$,

prove
$$\frac{ds}{dx} = \sqrt{\frac{2a}{x}}.$$

7. In the curve $y = a \log \sec \frac{x}{a}$,

prove
$$\frac{ds}{dx} = \sec\frac{x}{a},\quad \frac{ds}{dy} = \operatorname{cosec}\frac{x}{a},\quad \text{and } x = a\psi.$$

8. Show that the portion of the tangent to the curve

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

which is intercepted between the axes, is of constant length.

Find the area of the portion included between the axes and the tangent.

9. Find for what value of n the length of the subnormal of the curve $xy^n=a^{n+1}$ is constant. Also for what value of n the area of the triangle included between the axes and any tangent is constant.

10. Prove that for the catenary $y=c\cosh\frac{x}{c}$, the length of the perpendicular from the foot of the ordinate on the tangent is of constant length.

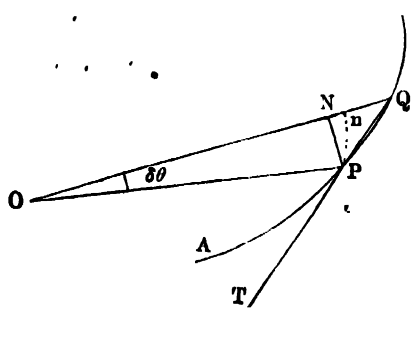
11. In the tractory

$$x=\sqrt{c^2-y^2}+\frac{c}{2}\log\frac{c-\sqrt{c^2-y^2}}{c+\sqrt{c^2-y^2}},$$

prove that the portion of the tangent intercepted between the point of contact and the axis of x is of constant length.

91. Polar Co-ordinates.

If the equation of the curve be referred to polar co-ordinates, suppose O to be the pole and P, Q two contiguous points on the curve. Let the co-ordinates of P and Q be (r, θ) and $(r+\delta r, \theta+\delta\theta)$ respectively. Let PN be the perpendicular on OQ, then NQ differs from



δr and NP from $r\delta\theta$ *by a quantity of higher order of smallness than* $\delta\theta$. (Art. 79.)

Let the arc measured from some fixed point A to P be called s and from A to Q, $s+\delta s$. Then arc $PQ=\delta s$. Hence, rejecting infinitesimals of order higher than the second, we have

$$\delta s^2 = (\text{chord } PQ)^2 = (NQ^2+PN^2) = (\delta r^2 + r^2\delta\theta^2),$$

and therefore

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1, \text{ or } \left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2,$$

or
$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

according as we divide by δs^2, δr^2, or $\delta\theta^2$ before proceeding to the limit.

92. Inclination of the Radius Vector to the Tangent.

Next, let ϕ be the angle which the tangent at any point P makes with the radius vector, then

$$\tan\phi = r\frac{d\theta}{dr}, \quad \cos\phi = \frac{dr}{ds}, \quad \sin\phi = \frac{rd\theta}{ds}.$$

For, with the figure of the preceding article, since, when Q has moved along the curve so near to P that Q and P may be considered as ultimately coincident, QP becomes the tangent at P and the angles OQT and OPT are each of them ultimately equal to ϕ, and

$$\tan\phi = Lt \tan NQP = Lt\frac{NP}{QN} = Lt\frac{r\delta\theta}{\delta r} = r\frac{d\theta}{dr};$$

$$\cos\phi = Lt\cos NQP = Lt\frac{NQ}{\text{chord } QP}$$

$$= Lt\frac{NQ}{\text{arc } QP} = Lt\frac{\delta r}{\delta s} = \frac{dr}{ds};$$

$$\sin\phi = Lt \sin NQP = Lt \frac{NP}{\text{chord } QP}$$
$$= Lt \frac{NP}{\text{arc } QP} = Lt \frac{r\delta\theta}{\delta s} = \frac{rd\theta}{ds}.$$

Ex. Find the angle ϕ in the case of the curve

$$r^n = a^n \sec(n\theta + a),$$

and prove that this curve is intersected by the curve

$$r^n = b^n \sec(n\theta + \beta)$$

at an angle which is independent of a and b. [I. C. S., 1886.]

Taking the logarithmic differential,

$$\frac{1}{r}\frac{dr}{d\theta} = \tan(n\theta + a),$$

whence $$\frac{\pi}{2} - \phi = n\theta + a.$$

In a similar manner for the second curve

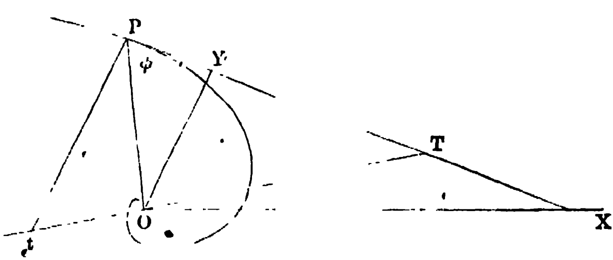
$$\frac{\pi}{2} - \phi' = n\theta + \beta,$$

ϕ' being the angle which the radius vector makes with the tangent to the second curve. Hence the angle between the tangents at the point of intersection is $a \sim \beta$.

93. Polar Subtangent, Subnormal.

Let OY be the perpendicular from the origin on the tangent at P.

Let TOt be drawn through O perpendicular to OP



and cutting the tangent in T and the normal in t. Then

OT is called the "*Polar Subtangent*" and Ot is called the "*Polar Subnormal.*"

It is clear that

$$OT = OP \tan\phi = r^2 \frac{d\theta}{dr} \quad\text{..............(1)},$$

and that
$$Ot = OP \cot\phi = \frac{dr}{d\theta} \quad\text{..................(2)}.$$

94. It is often found convenient when using polar co-ordinates to write $\frac{1}{u}$ for r, and therefore $-\frac{1}{u^2}\frac{du}{d\theta}$ for $\frac{dr}{d\theta}$. With this notation,

$$\text{Polar Subtangent} = r^2 \frac{d\theta}{dr} = -\frac{d\theta}{du}.$$

Ex. In the conic $\quad lu = 1 + e\cos\theta$

we have $\quad l = -e\sin\theta \frac{d\theta}{du}.$

Thus the length of the polar subtangent is $l/e \sin\theta$.

Also, from the figure, the angular co-ordinate of its extremity is

$$\theta - \frac{\pi}{2}.$$

Hence the co-ordinates of $T\,(r_1, \theta_1)$ satisfy the equation

$$r_1 = l/e \sin\left(\frac{\pi}{2} + \theta_1\right).$$

The locus of the extremity is therefore

$$lu = e\cos\theta;$$

that is, the directrix corresponding to that focus which is taken as origin.

95. Perpendicular from Pole on Tangent.

Let $\quad OY = p.$

Then $\quad p = r\sin\phi,$

and therefore

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\};$$

therefore
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad\qquad (1)$$
$$= u^2 + \left(\frac{du}{d\theta}\right)^2 \qquad\qquad (2).$$

Ex. In the spiral $\qquad r = a\dfrac{\theta^2}{\theta^2 - 1}$

we have $\qquad au = 1 - \theta^{-2},$

whence $\qquad a\dfrac{du}{d\theta} = 2\theta^{-3};$

and therefore, squaring and adding,

$$\frac{a^2}{p^2} = 1 - 2\theta^{-2} + \theta^{-4} + 4\theta^{-6}.$$

Thus, corresponding to $\theta = \pm 1$, we have

$$\frac{a^2}{p^2} = 4 \text{ and } p = \pm\frac{a}{2}.$$

96. The Pedal Equation.

The relation between p and r often forms a very convenient equation to the curve. It is called the Pedal equation.

(1) If the curve be given in Cartesians,

say $F(x, y) = 0$ .... (1),

the tangent is

$$XF_x + YF_y + ZF_z = 0$$

and
$$p^2 = \frac{F_z^2}{F_x^2 + F_y^2} \qquad\qquad (2).$$

If x, y be eliminated between equations (1), (2) and

$$x^2 + y^2 = r^2 \qquad\qquad (3),$$

the required equation will result.

Ex. If $x^2+y^2=2ax$,

$$X(x-a)+Yy=ax$$

is the equation of the tangent, and

$$p^2=\frac{a^2x^2}{(x-a)^2+y^2}=\frac{1}{4}\frac{r^4}{a^2},$$

or

$$r^2=2ap.$$

This result will also be evident geometrically.

(2) If the curve be given in Polars we may first obtain p in terms of r and θ by Art. 95, and then eliminate θ between this result and the equation to the curve.

Ex. Required the pedal equation of $r^m=a^m\sin m\theta$.

By logarithmic differentiation,

$$\frac{m}{r}\frac{dr}{d\theta}=m\cot m\theta,$$

$$\therefore \cot\phi=\cot m\theta \text{ or } \phi=m\theta,$$

whence

$$p=r\sin\phi=r\sin m\theta=r\frac{r^m}{a^m},$$

or

$$pa^m=r^{m+1}.$$

EXAMPLES.

1. In the equiangular spiral $r=ae^{\theta\cot a}$, prove

$$\frac{dr}{ds}=\cos a \text{ and } p=r\sin a.$$

2. For the involute of a circle, viz.,

$$\theta=\frac{\sqrt{r^2-a^2}}{a}-\cos^{-1}\frac{a}{r},$$

prove

$$\cos\phi=\frac{a}{r}.$$

3. In the parabola $\frac{2a}{r}=1-\cos\theta$, prove the following results:—

$$(a)\quad \phi=\pi-\frac{\theta}{2}.$$

(β) $p = \dfrac{a}{\sin \dfrac{\theta}{2}}$.

(γ) $p^2 = ar$.

(δ) Polar subtangent $= 2a \operatorname{cosec} \theta$.

4. For the cardioide $r = a(1 - \cos\theta)$, prove

(a) $\phi = \dfrac{\theta}{2}$.

(β) $p = 2a \sin^3 \dfrac{\theta}{2}$.

(γ) $p^2 = \dfrac{r^3}{2a}$.

(δ) Polar subtangent $= 2a \dfrac{\sin^3 \dfrac{\theta}{2}}{\cos \dfrac{\theta}{2}}$.

97. Maximum number of tangents from a point to a curve of the n^{th} degree.

Let the equation of the curve be $f(x, y) = 0$. The equation of the tangent at the point (x, y) is

$$Xf_x + Yf_y + Zf_z = 0,$$

where z is to be put equal to unity after the differentiation is performed. If this pass through the point h, k we have

$$hf_x + kf_y + f_z = 0.$$

This is an equation of the $(n-1)^{\text{th}}$ degree in x and y and represents a curve of the $(n-1)^{\text{th}}$ degree *passing through the points of contact* of the tangents drawn from the point (h, k) to the curve $f(x, y) = 0$. These two curves have $n(n-1)$ points of intersection, and therefore there are $n(n-1)$ *points of contact* corresponding to $n(n-1)$ *tangents, real or imaginary*, which can be drawn from a given point to a curve of the n^{th} degree.

Thus for a conic, a cubic, a quartic, the maximum number of tangents which can be drawn from a given point is 2, 6, 12 respectively.

98. Number of Normals which can be drawn to a Curve to pass through a given point.

Let h, k be the point through which the normals are to pass.

The equation of the normal to the curve $f(x, y) = 0$ at the point (x, y) is

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y}.$$

If this pass through h, k,

$$(h - x) f_y = (k - y) f_x.$$

This equation is of the n^{th} degree in x and y and represents a curve which goes *through the feet of all normals* which can be drawn from the point h, k to the curve. Combining this with $f(x, y) = 0$, which is also of the n^{th} degree, it appears that there are n^2 points of intersection, and that therefore there can be n^2 *normals, real or imaginary*, drawn to a given curve to pass through a given point.

For example, if the curve be an ellipse, $n=2$, and the number of normals is 4. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the curve, then

$$(h - x) \frac{y}{b^2} = (k - y) \frac{x}{a^2}$$

is the curve which, with the ellipse, determines the feet of the normals drawn from the point (h, k). This is a rectangular hyperbola which passes through the origin and through the point (h, k).

The student should consider how it is that an *infinite* number of normals can be drawn from the centre of a circle to the circumference.

99. The curves

$$(h - x) f_x + (k - y) f_y = 0 \text{..........(1)},$$

and $$(h - x) f_y - (k - y) f_x = 0 \text{..........(2)},$$

on which lie the points of contact of tangents and the feet of the normals respectively, which can be drawn to the curve $f(x, y) = 0$ so as to pass through the point (h, k), are the same for the curve $f(x, y) = a$. And, as equations (1) and (2) do not depend on a, they represent *the loci of the points of contact and of the feet of the normals* respectively for all values of a, that is, for all members of the family of curves obtained by varying a in $f(x, y) = a$ in any manner.

EXAMPLES.

1. Through the point h, k tangents are drawn to the curve

$$Ax^3 + By^3 = 1;$$

show that the points of contact lie on a conic.

2. If from any point P normals be drawn to the curve whose equation is $y^m = max^n$, show that the feet of the normals lie on a conic of which the straight line joining P to the origin is a diameter. Find the position of the axes of this conic.

3. The points of contact of tangents from the point h, k to the curve $x^3 + y^3 = 3axy$ lie on a conic which passes through the origin.

4. Through a given point h, k tangents are drawn to curves where the ordinate varies as the cube of the abscissa. Show that the locus of the points of contact is the rectangular hyperbola

$$2xy + kx - 3hy = 0,$$

and the locus of the remaining point in which each tangent cuts the curve is the rectangular hyperbola

$$xy - 4kx + 3hy = 0.$$

EXAMPLES.

1. Find the points on the curve

$$y = (x - 1)(x - 2)(x - 3)$$

at which the tangent is parallel to the axis of x.

Show also that the tangents at the first and third intersections with the x-axis are parallel, and at the middle intersection the tangent makes an angle 135° with that axis.

2. In any Cartesian curve the rectangle contained by the subtangent and the subnormal is equal to the square on the corresponding ordinate.

3. Show that the only Cartesian locus in which the ratio of the subtangent to the subnormal is constant is a straight line.

4. If the ratio of the subnormal to the subtangent vary as the square of the abscissa the curve is a parabola.

5. Show that in any curve

$$\frac{\text{Subnormal}}{\text{Subtangent}} = \left(\frac{\text{Normal}}{\text{Tangent}}\right)^2.$$

6. Find that normal to

$$\sqrt{xy} = a + x,$$

which makes equal intercepts upon the co-ordinate axes.

7. Prove that the sum of the intercepts of the tangent to

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

upon the co-ordinate axes is constant.

8. Show that in the curve

$$y = a \log (x^2 - a^2),$$

the sum of the tangent and the subtangent varies as the product of the co-ordinates of the point.

9. Show that in the curve

$$x^{m+n} = a^{m-n} y^{2n},$$

the m^{th} power of the subtangent varies as the n^{th} power of the subnormal.

10. In the curve $y^n = a^{n-1}x$ the subnormal $\propto \frac{y^2}{x}$ and the subtangent $\propto x$.

11. Show that in the curve $y = be^{-\frac{a}{x}}$ the subtangent varies as the square of the abscissa.

12. If in a curve the normal varies as the cube of the ordinate, find the subtangent and the subnormal.

13. Show that in the curve for which

$$s = c \log \frac{c}{y}$$

the tangent is of constant length.

14. Show that in the curve for which

$$y^2 = c^2 + s^2, \quad \text{(The Catenary)}$$

the perpendicular from the foot of the ordinate upon the tangent is of constant length.

15. Show that the polar subtangent in the curve $r = a\theta$ (The Spiral of Archimedes) varies as the square of the radius vector, and the polar subnormal is constant.

16. Show that the polar subtangent is constant in the curve

$$r\theta = a. \quad \text{(The Reciprocal Spiral.)}$$

17. Show that in the curve

$$r = ae^{\theta \cot \alpha}. \quad \text{(The Equiangular Spiral.)}$$

(1) the tangent makes a constant angle with the radius vector;

(2) the Polar Subtangent $= r \tan \alpha$;
the Polar Subnormal $= r \cot \alpha$;

(3) the loci of the extremities of the polar subtangent, the polar subnormal, the perpendicular upon the tangent from the pole are curves of the same species as the original.

18. Show that each of the several classes of curves (Cotes's Spirals)

$$r = ae^{m\theta}, \quad r\theta = a, \quad r \sin n\theta = a, \quad r \sinh n\theta = a,$$
$$r \cosh n\theta = a,$$

have pedal equations of the form

$$\frac{1}{p^2} = \frac{A}{r^2} + B,$$

where A and B are certain constants.

19. Find the angle of intersection of the Cardioides

$$r = a(1 + \cos\theta),$$
$$r = b(1 - \cos\theta).$$

20. Find the angle of intersection of

$$\left.\begin{aligned} x^2 - y^2 &= a^2 \\ x^2 + y^2 &= a^2\sqrt{2} \end{aligned}\right\}$$

21. Show that the condition of tangency of

$$x\cos\alpha + y\sin\alpha = p,$$

with
$$x^m y^n = a^{m+n},$$

is
$$p^{m+n}\,.\,m^m\,.\,n^n = (m+n)^{m+n} a^{m+n} \cos^m\alpha \sin^n\alpha.$$

Hence write down the equation of the locus of the foot of the perpendicular from the origin upon a tangent.

22. Show that in the curve (the cycloid)

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta),$$

$$\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2} \text{ and } \frac{ds}{dy} = \sqrt{2a/y}.$$

23. Show that in the curve (an epicycloid)

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta$$

we have

$$p = (a+2b)\sin\frac{a}{2b}\theta\,; \qquad \psi = \frac{a+2b}{2b}\theta\,; \qquad p = (a+2b)\sin\frac{a\psi}{a+2b}\,;$$

and that the pedal equation is

$$r^2 = a^2 + 4\frac{(a+b)b}{(a+2b)^2}p^2.$$

24. Show that the normal to $y^2 = 4ax$ touches the curve

$$27ay^2 = 4(x-2a)^3.$$

25. Show that the locus of the extremity of the polar subtangent of the curve

$$u = f(\theta),$$

$$u + f'\left(\frac{\pi}{2} + \theta\right) = 0$$

26. Show that the locus of the extremity of the polar subnormal of the curve

$$r=f(\theta),$$

is

$$r=f'\left(\theta-\frac{\pi}{2}\right).$$

27. In the curve

$$r\left(m+n\tan\frac{\theta}{2}\right)=1+\tan\frac{\theta}{2},$$

show that the locus of the extremity of the polar subtangent is

$$\frac{m-n}{2}r=1+\cos\theta.$$

CHAPTER IX.

ASYMPTOTES.

100. DEF. If a straight line cut a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity, it is called an asymptote to the curve.

101. To obtain the Asymptotes.

If
$$\phi(x, y) = 0 \qquad (1)$$
be the equation of any rational algebraic curve of the n^{th} degree, and
$$y = mx + c \qquad (2)$$
that of any straight line, the equation
$$\phi(x, mx + c) = 0 \qquad (3)$$
obtained by substituting the expression $mx + c$ for y gives the abscissae of the points of intersection.

This equation is in general of the n^{th} degree, showing that a curve of the n^{th} degree is in general cut in n points real or imaginary by any straight line.

The two constants of the straight line, viz. m and c, are at our choice. We are to choose them so as to make two of the roots of equation (3) infinite. We then have a line cutting the given curve so that two of the points of intersection are at an infinite distance from the origin.

Imagine equation (3) expanded out and expressed in descending powers of x as

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + K = 0 \quad \dots\dots(4),$$

A, B, C, etc. being certain functions of m and c.

The equation whose roots are the reciprocals of the roots of this equation is

$$A + Bz + Cz^2 + \dots + Kz^n = 0$$

$$\left(\text{by putting } x = \frac{1}{z}\right);$$

and it is evident that if A and B be both zero two roots of this equation for z will become evanescent, and therefore two roots of the equation for x become infinite. If then we choose m and c to satisfy the equations

$$A = 0, \quad B = 0,$$

and substitute their values in the equation

$$y = mx + c,$$

we shall obtain the equation of an asymptote.

102. It will be found in examples (and it admits of general proof) that the equation $A = 0$ contains m only and in a degree not higher than n. Also that $B = 0$ contains c in the first degree. Hence a curve of the n^{th} degree does not possess more than n asymptotes.

Ex. Find the asymptotes of the curve

$$y^3 - x^2y + 2y^2 + 4y + x = 0.$$

Putting $y = mx + c$,

$$(mx + c)^3 - x^2(mx + c) + 2(mx + c)^2 + 4(mx + c) + x = 0,$$

or $$(m^3 - m)x^3 + (3m^2c - c + 2m^2)x^2 + \dots \text{ etc.} = 0.$$

We now are to choose m and c so that

$$\left.\begin{aligned} m^3 - m &= 0 \\ \text{and} \quad 3m^2c - c + 2m^2 &= 0 \end{aligned}\right\}.$$

The first equation is a *cubic* for m and gives $m = 0$, 1 or -1.

The second equation is of the *first degree in c* and gives

$$c=\frac{2m^2}{1-3m^2}.$$

If $m=0$ we have $\qquad c=0;$

if $\quad m=1$ we have $\qquad c=-1;$

if $\quad m=-1$ we have $\qquad c=-1.$

Hence we obtain *three* asymptotes, viz.

$$y=0,$$
$$y=x-1,$$
$$y=-x-1.$$

EXAMPLES.

Find the asymptotes of

1. $y^3-6xy^2+11x^2y-6x^3+x+y=0.$

2. $y^3-4x^2y-xy^2+4x^3+4xy-4x^2=5.$

3. $y^3-3x^2y+xy^2-3x^3+2y^2+2xy+4x+5y+6=0.$

4. $(y+x+1)(y+2x+2)(y+3x+3)(y-x)+x^2+y^2-2=0.$

5. $(2x+3y)(3x+4y)(4x+5y)+26x^2+70xy+47y^2+2x+3y=1.$

103. The case of parallel Asymptotes.

After having formed equation (4) of Art. 101 by substitution of $mx+c$ for y and rearrangement, it sometimes happens that one or more of the values of m, deduced from the equation $A=0$, will make B vanish *identically*, and therefore *any* value of c will give a line cutting the curve in two points at infinity. In this case as the letter c is still at our choice, it may be chosen so as to make the third coefficient C vanish. It will be seen from examples that each such value of m now gives rise to two values of c. This is the case of parallel asymptotes. The two lines thus obtained each cut the curve at three points at infinity.

Ex. Find the asymptotes of the cubic curve,

$$y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x = 1.$$

Putting $mx + c$ for y and rearranging,

$$(m^3 - 5m^2 + 8m - 4)\,x^3 + (3m^2c - 10mc + 8c - 3m^2 + 9m - 6)\,x^2$$
$$+ (3mc^2 - 5c^2 - 6mc + 9c + 2m - 2)\,x + c^3 - 3c^2 + 2c - 1 = 0.$$

Choosing and
$$\left.\begin{array}{r} m^3 - 5m^2 + 8m - 4 = 0 \\ 3m^2c - 10mc + 8c - 3m^2 + 9m - 6 = 0 \end{array}\right\},$$

the first gives $(m-1)(m-2)^2 = 0,$

whence $m = 1$, 2 or 2.

If $m = 1$ the second equation gives $c = 0$ and the corresponding asymptote is $y = x$.

If $m = 2$ we have $12c - 20c + 8c - 12 + 18 - 6$ which vanishes identically for all finite values of c. Thus *any* line parallel to $y = 2x$ will cut the curve in two points at infinity. We may however choose c so that the next coefficient

$$3mc^2 - 5c^2 - 6mc + 9c + 2m - 2$$

vanishes for the value $m = 2$, giving

$$c^2 - 3c + 2 = 0, \text{ i.e. } c = 1 \text{ or } 2.$$

Thus each of the system of lines parallel to $y = 2x$ cuts the curve in *two* points at infinity. But of all this infinite system of parallel straight lines the two whose equations are

$$y = 2x + 1,$$

and

$$y = 2x + 2,$$

are the only ones which cut the curve in *three* points at infinity and therefore the name *asymptote* is confined to them.

The asymptotes are therefore

$$\left.\begin{array}{l} y = x \\ y = 2x + 1 \\ y = 2x + 2 \end{array}\right\}.$$

EXAMPLES.

Find the asymptotes of

1. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1.$
2. $y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 = 1.$
3. $(y^2 - x^2)^2 - 2(x^2 + y^2) = 1.$

104. Those asymptotes which are parallel to the y-axis will not be discovered by the above processes for their equations are of the form $x = a$, and are not included in the form $y = mx + c$ for a finite value of m. We, therefore, specially consider the case of those asymptotes which may be parallel to one or other of the co-ordinate axes.

105. Asymptotes Parallel to the Axes.

Let the equation of the curve be

$$\begin{aligned} a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n \\ + b_1x^{n-1} + b_2x^{n-2}y + \ldots\ldots\ldots\ldots + b_ny^{n-1} \\ + c_2x^{n-2} + \ldots \\ + \ldots = 0 \quad \ldots\ldots\ldots\ldots(1). \end{aligned}$$

If arranged in descending powers of x this is

$$a_0x^n + (a_1y + b_1)\,x^{n-1} + \ldots = 0 \;\ldots\ldots\ldots\ldots(2).$$

Hence, if a_0 vanish, and y be so chosen that

$$a_1y + b_1 = 0,$$

the coefficients of the two highest powers of x in equation (2) vanish, and therefore *two of its roots are infinite.* Hence the straight line $a_1y + b_1 = 0$ is an asymptote.

In the same way, if $a_n = 0$, $a_{n-1}x + b_n = 0$ is an asymptote.

Again, if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$, and if y be so chosen that

$$a_2y^2 + b_2y + c_2 = 0,$$

three roots of equation (2) become infinite, and the lines represented by

$$a_2y^2 + b_2y + c_2 = 0$$

represent a pair of asymptotes, real or imaginary, parallel to the axis of x.

Hence the rule to find those asymptotes which are parallel to the axes is, "*equate to zero the coefficients of the highest powers of x and y.*"

Ex. 1. Find the asymptotes of the curve

$$x^2y^2-x^2y-xy^2+x+y+1=0.$$

Here the coefficient of x^2 is y^2-y and the coefficient of y^2 is x^2-x. Hence $x=0$, $x=1$, $y=0$, and $y=1$ are asymptotes. Also, since the curve is one of the fourth degree, we have thus obtained all the asymptotes.

Ex. 2. Find the asymptotes of the cubic curve

$$x^3+2x^2y+xy^2-x^2-xy+2=0.$$

Equating to zero the coefficient of y^2 we obtain $x=0$, the only asymptote parallel to either axis.

Putting $mx+c$ for y,

$$x^3+2x^2(mx+c)+x(mx+c)^2-x^2-x(mx+c)+2=0,$$

or rearranging,

$$x^3(1+2m+m^2)+x^2(2c+2mc-1-m)+x(c^2-c)+2=0,$$

$1+2m+m^2=0$ gives two roots $m=-1$. $2c+2mc-1-m=0$ is an identity if $m=-1$ and this fails to find c. Proceeding to the next coefficient $c^2-c=0$ gives $c=0$ or 1.

Hence the three asymptotes are $x=0$, and the pair of parallel lines

$$y+x=0,$$
$$y+x=1.$$

EXAMPLES.

1. The asymptotes of $y^2(x^2-a^2)=x$ are

$$\left.\begin{array}{l} y=0 \\ x=\pm a \end{array}\right\}.$$

2. The co-ordinate axes are the asymptotes of

$$xy^3+x^3y=a^4.$$

3. The asymptotes of the curve $x^2y^2=c^2(x^2+y^2)$ are the sides of a square.

106. The methods given above will obtain all linear asymptotes. It is often more expeditious however to

obtain the oblique asymptotes as an approximation of the curve to a linear form at infinity as described in the next article.

107. Form of the Curve at Infinity. Another Method for Oblique Asymptotes.

Let P_r, F_r be used to denote rational algebraical expressions which contain terms of the r^{th} and lower, but of no higher degrees.

Suppose the equation of a curve of the n^{th} degree to be thrown into the form

$$(ax+by+c)\,P_{n-1}+F_{n-1}=0\ldots\ldots\ldots\ldots(1).$$

Then *any* straight line parallel to $ax+by=0$ obviously cuts the curve in *one* point at infinity; and to find the particular member of this family of parallel straight lines which cuts the curve in a second point at infinity, let us examine what is the ultimate linear form to which the curve gradually approximates as we travel to infinity in the above direction, thus obtaining the ultimate direction of the curve and forming the equation of the tangent at infinity. To do this we make the x and y of the curve become *large in the ratio given by*

$$x : y = -b : a,$$

and we obtain the equation

$$ax+by+c+Lt_{y=-\frac{a}{b}x=\infty}\left(\frac{F_{n-1}}{P_{n-1}}\right)=0.$$

If this limit be finite we have arrived at the equation of a straight line which at infinity represents the limiting form of the curve, and which satisfies the definition of an asymptote.

To obtain the value of the limit it is advantageous to put $x=-\dfrac{b}{t}$ and $y=\dfrac{a}{t}$, and then after simplification make $t=0$.

Ex. Find the asymptote of

$$x^3+3x^2y+3xy^2+2y^3=x^2+y^2+x.$$

We may write this curve as

$$(x+2y)(x^2+xy+y^2)=x^2+y^2+x,$$

whence the equation of the asymptote is given by

$$x+2y=Lt_{x=-2y\to\infty}\frac{x^2+y^2+x}{x^2+xy+y^2},$$

and putting $x=\dfrac{-2}{t}$, $y=\dfrac{1}{t}$ we have

$$x+2y=Lt_{t=0}\frac{\dfrac{4}{t^2}+\dfrac{1}{t^2}-\dfrac{2}{t}}{\dfrac{4}{t^2}-\dfrac{2}{t^2}+\dfrac{1}{t^2}}=Lt_{t=0}\frac{5-2t}{3}=\frac{5}{3},$$

i.e.,

$$x+2y=\frac{5}{3}.$$

EXAMPLE. Show that $x+y=\dfrac{a}{2}$ is the only real asymptote of the curve $(x+y)(x^4+y^4)=a(x^4+a^4)$.

108. Next, suppose the equation of a curve put into the form

$$(ax+by+c)F_{n-1}+F_{n-2}=0,$$

then the line $ax+by+c=0$ cuts the curve in two points at infinity, for no terms of the n^{th} or $(n-1)^{\text{th}}$ degrees remain in the equation determining the points of intersection. Hence in general the line

$$ax+by+c=0$$

is an asymptote. We say, *in general*, because if F_{n-1} be of the form $(ax+by+c)P_{n-2}$, itself containing a factor $ax+by+c$, there will be a *pair of asymptotes* parallel to $ax+by+c=0$, each cutting the curve in *three* points at infinity. The equation of the curve then becomes

$$(ax+by+c)^2P_{n-2}+F_{n-2}=0,$$

and the equations of the parallel asymptotes are

$$ax+by+c=\pm\sqrt{-Lt\frac{F_{n-2}}{P_{n-2}}},$$

where x and y in the limit on the right-hand side become infinite in the ratio $\frac{x}{y}=-\frac{b}{a}$.

Or, if the curve be written in the form

$$(ax+by)^2 P_{n-2}+(ax+by)F_{n-2}+f_{n-2}=0,$$

in proceeding to infinity in the direction $ax+by=0$, we have

$$(ax+by)^2+(ax+by)\,.\,Lt\frac{F_{n-2}}{P_{n-2}}+Lt\frac{f_{n-2}}{P_{n-2}}=0,$$

when the limits are to be obtained by putting $x=-\frac{b}{t}$, $y=\frac{a}{t}$, and then diminishing t indefinitely. We thus obtain a pair of parallel asymptotes,

$$ax+by=\alpha \text{ and } ax+by=\beta,$$

where α and β are the roots of

$$\rho^2+\rho Lt\frac{F_{n-2}}{P_{n-2}}+Lt\frac{f_{n-2}}{P_{n-2}}=0.$$

And other particular forms which the equation of the curve may assume can be treated similarly.

Ex. 1. To find the pair of parallel asymptotes of the curve

$$(2x-3y+1)^2(x+y)-8x+2y-9=0.$$

Here $$2x-3y+1=\pm\sqrt{Lt\frac{8x-2y+9}{x+y}},$$

where x and y become infinite in the direction of the line $2x=3y$.

Putting $x=\frac{3}{t}$, $y=\frac{2}{t}$, the right side becomes ± 2. Hence the asymptotes required are $2x-3y=1$ and $2x-3y+3=0$.

Ex. 2. Find the asymptotes of

$$(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0.$$

Here $$(x-y)^2-10(x-y)\,Lt_{x=y=\infty}\frac{x^2}{x^2+y^2}+12\,Lt_{x=y=\infty}\frac{y^2}{x^2+y^2}=0,$$

or $$(x-y)^2-5(x-y)+6=0,$$

giving the parallel asymptotes $x-y=2$ and $x-y=3$.

109. Asymptotes by Inspection.

It is now clear that if the equation $F_n=0$ break up into linear factors so as to represent a system of n straight lines, no two of which are parallel, they will be the asymptotes of any curve of the form

$$F_n+F_{n-2}=0.$$

Ex. 1. $$(x-y)(x+y)(x+2y-1)=3x+4y+5$$

is a cubic curve whose asymptotes are obviously

$$x-y=0,$$
$$x+y=0,$$
$$x+2y-1=0.$$

Ex. 2. $$(x-y)^2(x+2y-1)=3x+4y+5.$$

Here $x+2y-1=0$ is one asymptote. The other two asymptotes are parallel to $y=x$. Their equations are

$$x-y=\pm\sqrt{Lt_{t=0}\frac{3+4+5t}{1+2-t}}=\pm\sqrt{\frac{7}{3}}.$$

110. Case in which all the Asymptotes pass through the Origin.

If then, when the equation of a curve is arranged in homogeneous sets of terms, as

$$u_n+u_{n-2}+u_{n-3}+\ldots=0,$$

it be found that there are no terms of degree $n-1$, and if also u_n contain no repeated factor, the n straight lines passing through the origin, and whose equation is $u_n=0$, are the n asymptotes.

EXAMPLES.

Find the asymptotes of the following curves:

1. $y^3 = x^2(2a - x)$.
2. $y^3 = x(a^2 - x^2)$.
3. $x^3 + y^3 = a^3$.
4. $y(a^2 + x^2) = a^2x$.
5. $axy = x^3 - a^3$.
6. $y^2(2a - x) = x^3$.
7. $x^3 + y^3 = 3axy$.
8. $x^2y + y^2x = a^3$.
9. $x^2y^2 = (a + y)^2(b^2 - y^2)$.
10. $x^2y^2 = a^2y^2 - b^2x^2$.
11. $xy(x - y) - a(x^2 - y^2) = b^3$.
12. $(a^2 - x^2)y^2 = x^2(a^2 + x^2)$.
13. $xy^2 = 4a^2(2a - x)$.
14. $y^2(a - x) = x(b - x)^2$.
15. $x^2y = x^3 + x + y$.
16. $xy^2 + a^2y = x^3 + mx^2 + nx + p$.
17. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.
18. $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$.
19. $y(x - y)^3 = y(x - y) + 2$.
20. $x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y = 1$.
21. $(x + y)^2(x + 2y + 2) = x + 9y - 2$.
22. $3x^3 + 17x^2y + 21xy^2 - 9y^3 - 2ax^2 - 12axy - 18ay^2 - 3a^2x + a^2y = 0$.

111. Intersections of a Curve with its Asymptotes.

If a curve of the n^{th} degree have n asymptotes, no two of which are parallel, we have seen in Art. 109 that the equations of the asymptotes and of the curve may be respectively written

$$F_n = 0,$$

and

$$F_n + F_{n-2} = 0.$$

The n asymptotes therefore intersect the curve again at points lying upon the curve $F_{n-2} = 0$. Now each asymptote cuts its curve in two points at infinity, and therefore in $n - 2$ other points. Hence these $n(n - 2)$ points lie on a certain curve of degree $n - 2$. For example,

1. The asymptotes of a *cubic* will cut the curve again in *three points lying in a straight line;*
2. The asymptotes of a *quartic* curve will cut the curve again in *eight points lying on a conic section;*

and so on with curves of higher degree.

EXAMPLES.

1. Find the equation of a cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$, and which touches the axis of y at the origin, and goes through the point (3, 2).

2. Show that the asymptotes of the cubic

$$x^2y - xy^2 + xy + y^2 + x - y = 0$$

cut the curve again in three points which lie on the line

$$x + y = 0.$$

3. Find the equation of the conic on which lie the eight points of intersection of the quartic curve

$$xy(x^2 - y^2) + a^2y^2 + b^2x^2 = a^2b^2$$

with its asymptotes.

4. Show that the four asymptotes of the curve

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle.

112. Polar co-ordinates.

When the equation of a curve is given in the form

$$rf_1(\theta) + f_0(\theta) = 0 \dots\dots\dots\dots(1),$$

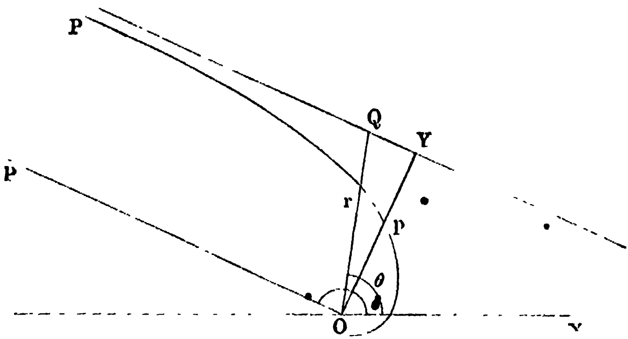
it is clear that the directions given by

$$f_1(\theta) = 0 \dots\dots\dots\dots(2),$$

are those in which r becomes infinite.

Let this equation be solved, and let the roots be α, β, γ, etc.

Let $\widehat{XOP} = \alpha$. Then the radius OP, the curve, and the asymptote meet at infinity towards P. Let $OY (= p)$



be the perpendicular upon the asymptote. Since OY is at right angles to OP it is the polar subtangent, and $p = -\frac{d\theta}{du}$. Let $\widehat{XOY} = \alpha'$, and let Q be any point whose co-ordinates are r θ upon the asymptote. Then the equation of the asymptote is

$$p = r \cos(\theta - \alpha') \qquad (3).$$

It is clear from the figure that $\alpha' = \alpha - \frac{\pi}{2}$.

To find the value of $-\frac{d\theta}{du}$ when $u = 0$, write $\frac{1}{u}$ for r in equation (1), and we have

$$f_1(\theta) + uf_0(\theta) = 0.$$

Whence differentiating

$$f_1'(\theta) + uf_0'(\theta) + \frac{du}{d\theta} f_0(\theta) = 0.$$

Putting $\theta = \alpha$, and therefore $u = 0$, we have (if $f_0'(\alpha)$ be finite)

$$\left(-\frac{d\theta}{du}\right)_{u=0} = \frac{f_0(\alpha)}{f_1'(\alpha)} \qquad\qquad (4).$$

Substitute this value of $\left(-\frac{d\theta}{du}\right)_{u=0}$ for p in equation (3) and we obtain

$$\frac{f_0(\alpha)}{f_1'(\alpha)} = r\cos\left(\theta - \alpha + \frac{\pi}{2}\right) = r\sin(\alpha - \theta).$$

Hence the equations of the asymptotes are

$$r\sin(\alpha - \theta) = \frac{f_0(\alpha)}{f_1'(\alpha)},$$

$$r\sin(\beta - \theta) = \frac{f_0(\beta)}{f_1'(\beta)},$$

etc.

113. Rule for Drawing the Asymptote.

After having found the value of $\left(-\frac{d\theta}{du}\right)_{u=0}$ imagine we stand at the origin looking in the direction of that value of θ which makes $u = 0$. Draw a line at right angles to that direction through the origin and of length equal to the calculated value of $\left(-\frac{d\theta}{du}\right)_{u=0}$ to the *right or to the left,* according as that value is *positive or negative.* Through the end of this line draw a perpendicular to it of indefinite length. This straight line will be the asymptote.

Ex. Find the asymptotes of the curve

$$r\cos\theta - a\sin\theta = 0.$$

Here $f_1(\theta)$ [illegible]